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Positive Solutions for a Class of p-Laplacian Systems with Sign-Changing Weight

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Abstract

We consider the system

$$\begin{cases} -\Delta_p u = \lambda F(x,u,v), & x \in \Omega, \\ -\Delta_q v = \lambda H(x,u,v), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases}$$

where $F(x,u,v) = [g(x)a(u) + f(v)]$, $H(x,u,v) = [g(x)b(v) + h(u)]$, Ω is a bounded domain in R^N $(N \geq 1)$ with smooth boundary $\partial\Omega$, λ is a real positive parameter and $\Delta_s z = div\,(|\nabla z|^{s-2}\nabla z)$, $s > 1$, $(s = p, q)$ is a s-laplacian operator. Here g is a C^1 sign-changing function that may be negative near the boundary and f, h, a, b are C^1 nondecreasing functions satisfying $a(0) \geq 0, b(0) \geq 0$,

$$\lim_{x\to\infty} \frac{a(x)}{x^{p-1}} = 0, \ \lim_{x\to\infty} \frac{b(x)}{x^{q-1}} = 0,$$

$$\lim_{x\to\infty} \ f(x) = \infty, \ \lim_{x\to\infty} \ h(x) = \infty,$$

and

$$\lim_{x\to\infty} \frac{f(M[h(x)]^{\frac{1}{q-1}})}{x^{p-1}} = 0, \forall M > 0,$$

by applying the method of sub-super solutions the existence of a positive solution is established for the above system.

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1 Introduction

Consider the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda F(x,u,v), & x \in \Omega, \\ -\Delta_q v = \lambda H(x,u,v), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in R^N $(N \geq 1)$ with smooth boundary $\partial\Omega$, Δ_s is the called s-Laplacian operator i.e. $\Delta_s z = div\,(|\nabla z|^{s-2}\nabla z)$, $s > 1$ $(s = p, q)$, λ is a positive parameter, $F(x,u,v) = [g(x)a(u)+f(v)]$, $H(x,u,v) = [g(x)b(v) + h(u)]$ and functions satisfying the following assumptions:

(H1) $f, h \in C^1([0,\infty))$ are nondecreasing, $f(x) \to \infty$, $h(x) \to \infty$ as $x \to \infty$, and $\lim\limits_{x\to\infty} \frac{f(M[h(x)]^{\frac{1}{q-1}})}{x^{p-1}} = 0$, $\forall M > 0$.

(H2) $f, h \in C^1([0,\infty))$ are non-negative, nondecreasing, and $\lim\limits_{x\to\infty} \frac{a(x)}{x^{p-1}} = 0$, $\lim\limits_{x\to\infty} \frac{b(x)}{x^{q-1}} = 0$.

In this paper, we focus on sign-changing weight functions g and also do not assume any sign condition on $f(0)$ or $h(0)$. Hence in our system $F(x,0,0)$ or $H(x,0,0)$ could be negative for some $x \in \Omega$.
To precisely state our theorem we first consider the eigenvalue problem

$$\begin{cases} -\Delta_s V = \lambda |V|^{s-2}V, & x \in \Omega, \\ V = 0, & x \in \partial\Omega,\ (\mathrm{s = p,q}), \end{cases} \tag{2}$$

let λ_1, λ_2 be the respective first eigenvalues of Δ_p, Δ_q with Dirichlet boundary conditions and ϕ_1, ϕ_2 the corresponding eigenfunctions with $\phi_1, \phi_2 > 0$ and $\|\phi_1\|_\infty = \|\phi_2\|_\infty = 1$. Hence there exist $\delta \geq 0$ and $\sigma_1, \sigma_2 \in (0,1]$ and $m > 0$ such that

$$\begin{cases} |\nabla\phi_1|^p - \lambda_1\phi_1^p \geq m & on\overline{\Omega}_\delta, \\ \phi_1 \geq \sigma_1 & on\Omega - \overline{\Omega}_\delta, \end{cases} \tag{3}$$

and

$$\begin{cases} |\nabla\phi_2|^q - \lambda_2\phi_2^q \geq m & on\overline{\Omega}_\delta, \\ \phi_2 \geq \sigma_2 & on\Omega - \overline{\Omega}_\delta, \end{cases} \tag{4}$$

where $\Omega_\delta = \{x \in \Omega | d(x,\partial\Omega) \leq \delta\}$.

Here we assume that the weight g takes negative values in Ω_δ but require g to be strictly positive in $\Omega - \overline{\Omega}_\delta$ then exist positive β, η constants such that $g(x) \geq -\beta$ on Ω_δ and $g(x) \geq \eta$ on $\Omega - \overline{\Omega}_\delta$. Let $s_0 \geq 0$ be such that $\eta a(s)+f(s) > 0$ and $\eta b(s)+h(s) > 0$ for every $s > s_0$, and $f_0 = max\{0, -f(0)\}$, $h_0 = max\{0, -h(0)\}$. for $\gamma > \frac{s_0}{\alpha}$ we define :

$$Q_1(\gamma) = min\{\frac{m\gamma}{\beta a((\frac{p-1}{p})\gamma^{\frac{1}{p-1}}) + f_0}, \frac{m\gamma}{\beta b((\frac{q-1}{q})\gamma^{\frac{1}{q-1}}) + h_0}\},$$

$$Q_2(\gamma) = max\{\frac{\gamma\lambda_1}{\eta a((\frac{p-1}{p})\gamma^{\frac{1}{p-1}}\alpha) + f((\frac{q-1}{q})\gamma^{\frac{1}{q-1}}\alpha)}, \frac{\gamma\lambda_2}{\eta a((\frac{q-1}{q})\gamma^{\frac{1}{q-1}}\alpha) + h((\frac{p-1}{p})\gamma^{\frac{1}{p-1}}\alpha)}\},$$

where $\alpha = min\{\sigma_1^{\frac{p}{p-1}}, \sigma_2^{\frac{q}{q-1}}\}$ and obtained the following theorem:

Theorem 1.1: Assume that $(H1) - (H2)$ hold and $G = \{s > \frac{s_0}{\alpha} : Q_2(s) \leq Q_1(s)\} \neq \emptyset$. Let $S = \bigcup_{s \in G}[Q_2(s), Q_1(s)]$, for $\lambda \in S$ system (1) has at least one positive solution.

2 Proof of Theorem 1.1

We shall establish Theorem by constructing a positive weak sub-solution (ψ_1, ψ_2) and super-solution (z_1, z_2) of (1) such that $\psi_i \leq z_i$ for $i = 1, 2$. that is, ψ_i satisfies:

$$\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla w dx \leq \lambda \int_\Omega (g(x)a(\psi_1) + f(\psi_2))w dx, \tag{5}$$

$$\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla w dx \leq \lambda \int_\Omega (g(x)b(\psi_2) + h(\psi_1))w dx, \tag{6}$$

and z_i satisfies:

$$\int_\Omega |\nabla z_1|^{p-2}\nabla z_1.\nabla w dx \geq \lambda \int_\Omega (g(x)a(z_1) + f(z_2))w dx, \tag{7}$$

$$\int_\Omega |\nabla z_2|^{q-2}\nabla z_2.\nabla w dx \geq \lambda \int_\Omega (g(x)b(z_2) + h(z_1))w dx, \tag{8}$$

for $w \in W = \{\phi \in C_0^\infty : \phi \geq 0 \ in \Omega\}$. let $\lambda \in S$ and $\gamma > \frac{s_0}{\alpha}$ be such that $\lambda \in [Q_2(s), Q_1(s)]$. We shall verify that $(\psi_1, \psi_2) = ((\frac{p-1}{p})\gamma^{\frac{1}{p-1}}\phi_1^{\frac{p}{p-1}}, (\frac{q-1}{q})\gamma^{\frac{1}{q-1}}\phi_2^{\frac{q}{q-1}})$ is a sub-solution.

$$\begin{aligned}
\int_{\Omega} |\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla w &= \gamma\int_{\Omega}\phi_1|\nabla\phi_1|^{p-2}\nabla\phi_1.\nabla w dx \\
&= \gamma\{\int_{\Omega}|\nabla\phi_1|^{p-2}\nabla\phi_1\nabla(\phi_1 w_1)dx - \int_{\Omega}|\nabla\phi_1|^p w dx\} \\
&= \gamma\int_{\Omega}(\lambda_1\phi_1^p - |\nabla\phi_1|^p)w dx.
\end{aligned}$$

First the case when $x \in \overline{\Omega}_\delta$, since $\lambda \le Q_1(\gamma)$, we have $\lambda \le \frac{m\gamma}{\beta a((\frac{p-1}{p})\gamma^{\frac{1}{p-1}})+f_0}$ then $-m\gamma \le \lambda[-\beta a((\frac{p-1}{p})\gamma^{\frac{1}{p-1}}) - f_0]$ hence

$$\begin{aligned}
\int_{\overline{\Omega}_\delta} |\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla w &= \gamma\int_{\overline{\Omega}_\delta}(\lambda_1\phi_1^p - |\nabla\phi_1|^p)w dx \\
&\le \gamma\int_{\overline{\Omega}_\delta}(-m)w dx = \int_{\overline{\Omega}_\delta}(-m\gamma)w dx \\
&\le \lambda\int_{\overline{\Omega}_\delta}[-\beta a((\frac{p-1}{p})\gamma^{\frac{1}{p-1}}) + f_0]w dx \\
&\le \lambda\int_{\overline{\Omega}_\delta}[g(x)a(\psi_1) + f(\psi_2)]w dx.
\end{aligned}$$

Next consider the case when $x \in \Omega - \overline{\Omega}_\delta$. Since $\lambda \ge Q_2(\gamma)$, we have $\lambda \ge \frac{\gamma\lambda_1}{\eta a((\frac{p-1}{p})\gamma^{\frac{1}{p-1}}\alpha)+f((\frac{q-1}{q})\gamma^{\frac{1}{q-1}}\alpha)}$, thus

$$\begin{aligned}
\int_{\Omega-\overline{\Omega}_\delta} |\nabla\psi_1|^{p-2}\nabla\psi_1.\nabla w &= \gamma\int_{\Omega-\overline{\Omega}_\delta}(\lambda_1\phi_1^p - |\nabla\phi_1|^p)w dx \\
&\le \gamma\int_{\Omega-\overline{\Omega}_\delta}\lambda_1 w dx = \int_{\Omega-\overline{\Omega}_\delta}\gamma\lambda_1 w dx \\
&\le \lambda\int_{\Omega-\overline{\Omega}_\delta}\eta a((\frac{p-1}{p})\gamma^{\frac{1}{p-1}}\alpha) + f((\frac{q-1}{q})\gamma^{\frac{1}{q-1}}\alpha)w dx \\
&\le \lambda\int_{\Omega-\overline{\Omega}_\delta}(g(x)a(\psi_1) + f(\psi_2))w dx.
\end{aligned}$$

Then proved (5).

Now we have for ψ_2 when $x \in \overline{\Omega}_\delta$: $\lambda \le Q_1(\gamma)$, thus $\lambda \le \frac{m\gamma}{\beta b((\frac{q-1}{q})\gamma^{\frac{1}{q-1}})+h_0}$, then $-m\gamma \le \lambda[-\beta b((\frac{q-1}{q})\gamma^{\frac{1}{q-1}}) - h_0]$, hence

$$\int_{\overline{\Omega}_\delta} |\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla w = \gamma\int_{\overline{\Omega}_\delta}(\lambda_2\phi_2^q - |\nabla\phi_2|^q)w dx$$

$$
\begin{aligned}
&\le \gamma\int_{\overline{\Omega}_\delta}(-m)wdx=\int_{\overline{\Omega}_\delta}(-m\gamma)wdx\\
&\le \lambda\int_{\overline{\Omega}_\delta}[-\beta b((\frac{q-1}{q})\gamma^{\frac{1}{q-1}})+h_0]wdx\\
&\le \lambda\int_{\overline{\Omega}_\delta}[g(x)b(\psi_2)+h(\psi_1)]wdx.
\end{aligned}
$$

and for $x\in\Omega-\overline{\Omega}_\delta$ we have $\lambda\ge Q_2(\gamma)$ then $\lambda\ge\frac{\gamma\lambda_2}{\eta b((\frac{q-1}{q})\gamma^{\frac{1}{q-1}}\alpha)+h((\frac{p-1}{p})\gamma^{\frac{1}{p-1}}\alpha)}$.

$$
\begin{aligned}
\int_{\Omega-\overline{\Omega}_\delta}|\nabla\psi_2|^{q-2}\nabla\psi_2.\nabla w &= \gamma\int_{\Omega-\overline{\Omega}_\delta}(\lambda_2\phi_2^q-|\nabla\phi_2|^q)wdx \le \int_{\Omega-\overline{\Omega}_\delta}\gamma\lambda_2 wdx\\
&\le \lambda\int_{\Omega-\overline{\Omega}_\delta}\eta b((\frac{q-1}{q})\gamma^{\frac{1}{q-1}}\alpha)+h((\frac{p-1}{p})\gamma^{\frac{1}{p-1}}\alpha)wdx\\
&\le \lambda\int_{\Omega-\overline{\Omega}_\delta}(g(x)b(\psi_2)+h(\psi_1))wdx.
\end{aligned}
$$

Then proved (6) and (ψ_1,ψ_2) is a sub-solution.

Now we proved for c large enough $(z_1,z_2)=(\frac{c}{\|e_p\|}\lambda^{\frac{1}{p-1}}e_p,\ [2h(c\lambda^{\frac{1}{p-1}})]^{\frac{1}{q-1}}\lambda^{\frac{1}{q-1}}e_q)$ is a super-solution where e_s is the unique positive solution of

$$
\begin{cases}
-\Delta_s e_s=1, & x\in\Omega,\\
e_s=0, & x\in\partial\Omega.\ for \text{ s = (p,q)},
\end{cases}
$$

we know a is p-sublinear and f,h satisfy $(H1)$ by choosing c large we have

$$
\frac{\|e_p\|_\infty^{p-1}[\|g\|_\infty a(c\lambda^{\frac{1}{p-1}})+f([2h(c\lambda^{\frac{1}{p-1}})]^{\frac{1}{q-1}}\lambda^{\frac{1}{q-1}}e_q)]}{c^{p-1}}\le 1,
$$

then

$$
\begin{aligned}
\int_\Omega|\nabla z_1|^{p-2}\nabla z_1.\nabla wdx &= \lambda(\frac{C}{\|e_p\|_\infty})^{p-1}\int_\Omega|\nabla e_p|^{p-2}\nabla e_p.\nabla wdx\\
&= \lambda(\frac{C}{\|e_p\|_\infty})^{p-1}\int_\Omega wdx\\
&\ge \lambda\int_\Omega[\|g\|_\infty a(c\lambda^{\frac{1}{p-1}})+f([2h(c\lambda^{\frac{1}{p-1}})]^{\frac{1}{q-1}}\lambda^{\frac{1}{q-1}}e_q)]\ wdx\\
&\ge \lambda\int_\Omega[g(x)a(c\lambda^{\frac{1}{p-1}}\frac{e_p}{\|e_p\|_\infty})+f(z_2)]wdx\\
&= \lambda\int_\Omega(g(x)a(z_1)+f(z_2))wdx,
\end{aligned}
$$

and similarly by c large we have

$$\frac{\|g\|_\infty b([2h(c\lambda^{\frac{1}{p-1}})]^{\frac{1}{q-1}}\lambda^{\frac{1}{q-1}}\, e_q)}{h(c\lambda^{\frac{1}{p-1}})} \leq 1,$$

then

$$\begin{aligned}
\int_\Omega |\nabla z_2|^{q-2}\nabla z_2.\nabla w dx &= 2\lambda\ h(c\lambda^{\frac{1}{p-1}})\int_\Omega w dx \\
&= \lambda\int_\Omega [h(c\lambda^{\frac{1}{p-1}}) + h(c\lambda^{\frac{1}{p-1}})]w dx \\
&\geq \lambda\int_\Omega [\|g\|_\infty b([2h(c\lambda^{\frac{1}{p-1}}) + h(c\lambda^{\frac{1}{p-1}})]w dx \\
&\geq \lambda\int_\Omega (g(x)b(z_2) + h(z_1))w dx.
\end{aligned}$$

Hence (z_1, z_2) is a super solution. by choosing c large enough we have $\psi_i \leq z_i$ for $i = 1, 2$. then there exits a positive solution (u, v) with $\psi_1 \leq u \leq z_1$, $\psi_2 \leq v \leq z_2$. this completes the proof of theorem. $\square$

References

[1] Hai D.D and R. Shivaji. An existence result on positive solutions for a class of p-Laplacian systems. Nonlin. Anal. 56(2004), 1007-1010.

[2] Jaffar Ali, R. Shivaji. Existence results for classes of Laplacian systems with sign-changing weight. Applied Mathematics Letters 20 (2007) 558-562.

[3] J. Ali, R. Shivaji, Positive solutions for a class of p-Laplacian systems with multiple parameters, J. Math. Anal. Appl. (2007).

[4] C. Maya, S. Oruganti and R. Shivaji, Positive solutions for classes of p-Laplacian Equations, Diff. Int. Eqns. Vol 16, number 6 , june 2003, pages 757 - 768

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