Abstract

If $T$ is an analytic function mapping the unit disk $D$ into itself, we define the composition operator $C_T$ on the space $H^2(\beta)$ by $C_T f = f \circ T$. In this paper, we investigate the relationship between properties of the symbol $T$ and the quasihyponormality of the operators $C_T$ and $C_T^*$. 

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1. PRELIMINARIES

Let $f$ be an analytic map on the open unit disk $D$ given by the Taylor’s series.

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

Let $\beta = \{ \beta_n \}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta_0 = 1$ and $
\frac{\beta_{n+1}}{\beta_n} \rightarrow 1$ as $n \rightarrow \infty$.

The set $H^2(\beta)$ of formal complex power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that

$$\| f \|_{\beta}^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$$

is a Hilbert space of functions analytic in the unit disc with the inner product.

$$\langle f, g \rangle_{\beta} = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2$$

for $f$ as above and $g (z) = \sum_{n=0}^{\infty} b_n z^n$.

Let $D$ be the open unit disk in the complex plane and let $T: D \rightarrow D$ be an analytic self-map of the unit disk and consider the corresponding composition operator $C_T$ acting on $H^2(\beta)$, i.e.,

$$C_T (f) = f \circ T, \quad f \in H^2(\beta)$$

The operators $C_T$ are not necessarily defined on all of $H^2(\beta)$. They are everywhere defined in some special cases: on the classical Hardy space $H^2$ (the case when $\beta_n = 1$ for all $n$). See for example [7], and on a general space $H^2 (\beta)$ if the function $T$ is analytic on some open set containing the closed unit disk having supremum norm strictly smaller than one (see [11]). There are a lot of other known properties of composition operators, on the classical Hardy space $H^2$ (See for example [1], [5] and [7]), and on more general space $H^2(\beta)$ (see [3], [4], [8], [10] and [11]).

In [2], Cowen’s and Kriete obtained a nice correlation between hyponormality of composition operators on $H^2$ and the Denjoy-Wolff point of the induced map.

In [9], Nina Zorboska obtained some results on the hyponormality of composition operators and their adjoints.

In this article, we are interested in the $M$-quasihyponormality of composition operators and their adjoints.

2. An operator $T$ on a Hilbert space $H$ is called $M$-quasihyponormal if there exists $M>0$ such that
If $M = 1$, $T$ is said to be quasihyponormal.

Furuta et al. [6], introduced a new class ‘class A’ operators as follows.

An operator $T$ belongs to class $A$ if and only if

$$(T^* | T^2)^{1/2} \geq T^*T$$

and showed that this class is included in the class of paranormal operators.

Let $\omega$ be a point on the open disk. Define

$$k_\omega^\beta(z) = \sum_{n=0}^{\infty} \frac{Z^n}{n^2}$$

Then the function $k_\omega^\beta$ is a point evaluation for $H^2(\beta)$.

Then $k_\omega^\beta$ is in $H^2(\beta)$ and $\|k_\omega^\beta\|^2 = \sum_{n=0}^{\infty} \frac{|\omega|^{2n}}{\beta_n^2}$.

Thus, $\|K_\omega\|$ is an increasing function of $|\omega|$.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$\langle f, k_\omega^\beta \rangle = \sum_{n=0}^{\infty} a_n \omega^n \beta_n^2$$

Therefore,

$$\langle f, k_\omega^\beta \rangle = f(\omega)$$

for all $f$ and $k_\omega^\beta$ is known as the point evaluation kernel at $\omega$.

It can be easily shown that

$$C^*_T k_\omega^\beta = k_\beta^\beta_{T(\omega)}$$

and $k_0^\beta = 1$ (the function identically equal to 1).

**Theorem 2.1:**

If $C_T$ is $M$-quasihyponormal then $\|k_{T(0)}^\beta\|^2 \leq M^2$

**Proof:**
C_T is M-quasihyponormal.

\[ \left\langle M^2 C_T^* C_T f, f \right\rangle - \left\langle (C_T^*)^2 f, f \right\rangle \geq 0, \quad \text{for all } f \in H^2(\beta). \]
\[ M^2 \left\langle C_T^2 f, C_T^2 f \right\rangle - \left\langle (C_T^*)^2 C_T C_T f, f \right\rangle \geq 0 \]
\[ M^2 \left\langle C_T^2 f, C_T^2 f \right\rangle - \left\langle C_T^* C_T^* f, C_T^* f \right\rangle \geq 0 \]
\[ M^2 \| C_T^2 f \|^2 \geq \| C_T^* C_T f \|^2 \]

Let \( f = k_0^\beta \), we have,
\[ M^2 \| C_T C_T k_0^\beta \|^2 \beta \geq \| C_T^* C_T k_0^\beta \|^2 \beta \]
\[ M^2 \| C_T k_0^\beta \|^2 \beta \geq \| C_T^* k_0^\beta \|^2 \beta \]
\[ M^2 \| k_0^\beta \|^2 \beta \geq \| k_{T(0)}^\beta \|^2 \beta \]
\[ \| k_{T(0)}^\beta \|^2 \beta \geq M^2. \quad [9] \]

**Theorem 2.2:**

A partial isometry composition operator \( C_T \) on \( H^2(\beta) \) is M-quasihyponormal then \( M^2 \geq 1 \).

**Proof:**

\( C_T \) is M-quasihyponormal.

\[ \| C_T^* C_T f \|^2 \leq M^2 \| C_T^2 f \|, \quad \text{for all } f \in H^2(\beta) \]
\[ M^2 C_T^2 C_T^2 - 2k(C_T^* C_T)^2 + k^2 \geq 0 \quad \text{for all } k > 0 \]
\[ (M^2 C_T^* C_T^2 - 2k(C_T^* C_T)^2 + k^2) \geq 0 \]
\[ M^2 C_T^* C_T C_T C_T C_T - 2k C_T^* C_T C_T C_T C_T C_T + k^2 C_T^* C_T C_T \geq 0 \]
\[ M^2 C_T^* C_T C_T - 2k C_T^* C_T C_T C_T + k^2 C_T^* C_T \geq 0 \]
\[ M^2 C_T^* C_T C_T - 2k C_T^* C_T C_T + k^2 C_T^* C_T \geq 0 \]
\[ M^2 \| C_T^2 f \|^2 - 2k \| C_T f \|^2 + k^2 \| C_T f \|^2 \geq 0 \]
Let $f = k_0^\beta$, we have,

\[
\begin{align*}
M^2 \| C_T^2 k_0^\beta \|_\beta^2 - 2k \| C_T k_0^\beta \|_\beta^2 + k^2 \| C_T k_0^\beta \|_\beta^2 & \geq 0 \\
M^2 \| C_T C_T k_0^\beta \|_\beta^2 - 2k \| k_0^\beta \|_\beta^2 + k^2 \| k_0^\beta \|_\beta^2 & \geq 0 \\
M^2 \| C_T k_0^\beta \|_\beta^2 - 2k + k^2 & \geq 0 \\
M^2 \| k_0^\beta \|_\beta^2 - 2k + k^2 & \geq 0 \\
M^2 - 2k + k^2 & \geq 0
\end{align*}
\]

By elementary properties of real quadratic form, we get, $M^2 \geq 1$.

**Theorem 2.3:**

If $C_T$ is hyponormal composition operator on $H^2(\beta)$ and $C_T^*$ is $M$-quasihyponormal then $M^2 \geq 1$.

**Proof:**

$C_T^*$ is $M$-quasihyponormal

\[
\begin{align*}
\left\langle M^2 C_T^2 C_T^{*2} f, f \right\rangle - \left\langle (C_T C_T^*)^2 f, f \right\rangle & \geq 0, \quad \text{for all } f \in H^2(\beta) \\
M^2 \left\langle C_T^{*2} f, f \right\rangle - \left\langle C_T C_T^* C_T f, f \right\rangle & \geq 0 \\
M^2 \left\langle C_T^{*2} f, C_T^{*2} f \right\rangle - \left\langle C_T C_T^* f, C_T C_T^* f \right\rangle & \geq 0 \\
M^2 \| C_T^{*2} f \|_\beta^2 - \| C_T C_T^* f \|_\beta^2 & \geq 0
\end{align*}
\]

Let $f = k_0^\beta$, we have,

\[
\begin{align*}
M^2 \| C_T^{*2} k_0^\beta \|_\beta^2 - \| C_T C_T^* k_0^\beta \|_\beta^2 & \geq 0 \\
M^2 \| C_T^{*2} k_0^\beta \|_\beta^2 & \geq \| C_T C_T^* k_0^\beta \|_\beta^2 \\
M^2 \| C_T^* k_{T(0)}^\beta \|_\beta^2 & \geq \| C_T k_0^\beta \|_\beta^2
\end{align*}
\]

Since $C_T$ is hyponormal, we have $T(0) = 0$. [9]

\[
M^2 \| C_T^* k_0^\beta \|_\beta^2 \geq \| C_T k_0^\beta \|_\beta^2
\]
Theorem 2.4:

If $C_T$ be quasihyponormal on the space $H^2(\beta)$, then $T(0) = 0$.

Proof:

Let $C_T$ be quasihyponormal on $H^2(\beta)$ and $k_0^{\beta}$ be point evaluation at 0.

Then $\left\| C_T^* C_T f \right\|_\beta \leq \left\| C_T^2 f \right\|_\beta$ for all $f$ in $H^2(\beta)$ and if $f = k_0^{\beta}$, we have,

\[
\left\| C_T^* C_T k_0^{\beta} \right\|_\beta^2 = \left\| C_T^* \right\|_\beta^2 \left\| k_0^{\beta} \right\|_\beta^2 = \left\| k_0^{T(0)} \right\|_\beta^2 = \sum_{n=0}^{\infty} \frac{1}{\beta_n^2} |T(0)|^{2n} \leq \left\| C_T^2 \right\|_\beta^2 \left\| k_0^{\beta} \right\|_\beta^2 = \left\| C_T^* C_T k_0^{\beta} \right\|_\beta^2 = \left\| k_0^{\beta} \right\|_\beta^2 = 1
\]

which implies, since $\beta_0 = 1$, that $T(0) = 0$.

Theorem 2.5:

If $C_T$ is of Class A on $H^2(\beta)$ then $T(0) = 0$.

Proof:

If $C_T$ is of Class A, implies

\[
(C_T^* C_T)^2 \leq C_T^* |C_T|^2 C_T
\]
\[
(C_T^* C_T)^2 \leq C_T^* (C_T^* C_T) C_T
\]
(C^*_T C_T)^2 \leq C^*_T C_T^2
\langle C^*_T C_T C^*_T C_T f, f \rangle \leq \langle C^*_T C_T^2 f, f \rangle \quad \text{for all } f \in H^2(\beta).
\langle C^*_T C_T f, C^*_T C_T f \rangle \leq \langle C^2_T f, C^2_T f \rangle
\| C^*_T C_T f \|^2_\beta \leq \| C^2_T f \|^2_\beta

Let f = k^\beta_0, we have,
\| C^*_T C_T k^\beta_0 \|^2_\beta \leq \| C^2_T k^\beta_0 \|^2_\beta
\| C^*_T k^\beta_0 \|^2_\beta \leq \| C_T C_T k^\beta_0 \|^2_\beta
\| k^\beta_{T(0)} \|^2_\beta \leq \| C_T k^\beta_0 \|^2_\beta = \| k^\beta_0 \|^2_\beta = 1
\| k^\beta_{T(0)} \|^2_\beta \leq 1, \text{ by theorem } 4, \text{ which implies that } T(0) = 0.

References


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