Traveling Wave Fronts in an Epidemic Model with Nonlocal Diffusion and Time Delay

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Abstract

This paper is concerned with the existence of traveling wave fronts of an epidemic model with nonlocal diffusion and delay. By constructing proper upper and lower solutions, we prove the existence of a traveling wave front which admits precisely asymptotic behavior.

Mathematics Subject Classification (2000): 35K57, 35R20, 92D25

Keywords: nonlocal diffusion, time delay, epidemic model

1 Introduction

In epidemic dynamics, one important model takes the following form [11]

\[
\frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2} - au(x,t) + bu(x,t-\tau)[1-u(x,t)]
\] (1.1)

where \(x \in \mathbb{R}, t > 0\) and all the parameters are positive. As pointed in Ruan and Xiao [11], Equation (1.1) is a host-vector model for a disease without immunity in which the current density of infectious vectors is related to the number of infectious hosts at earlier times. The traveling wave fronts of (1.1) have been investigated in recent years, see [8, 11].

During the development of delayed differential equations, one typical and important model is the following classical Hutchinson equation

\[
\frac{du(t)}{dt} = ru(t) \left[1 - \frac{u(t-\tau)}{K}\right]
\] (1.2)

in which all parameters are positive. In Hutchinson [6], the author interpreted the reason why a discrete time delay was incorporated into the Logistic model, and we refer to [5, 7, 12] for more details.
Motivated by the model (1.2), we further incorporate time delay into (1.1), then we get the following diffusion model with time delay

\[
\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} - au(x,t) + bu(x, t-\tau)[1 - u(x, t - \tau)]
\]

(1.3)

in which all the parameters are same to that in (1.1).

In the models (1.1) and (1.3), the diffusion of individuals is described by the classical Laplacian operator. However, the Laplacian operator is not sufficient accurate in describing some spatial-temporal evolutionary processes, for example, the embryological development case, and this was pointed by Murray [9]. In order to overcome the shortcoming of Laplacian operator, researchers introduced the so-called nonlocal diffusion evolutionary equation (or the integral-differential equation). Typical nonlocal diffusion evolutionary equation takes the form of

\[
\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} J(x-y)[u(y,t) - u(x,t)]dy + f(u(x,t)),
\]

(1.4)

where \(J(x)\) is a probability function describing the spatial diffusion of the individuals, and we can refer to Murray [9] for more details. For more backgrounds for such a model (1.4), we also can refer to [1] for material science, [13] for a neutral network model, [2] for a convolution model in phase transition and [4] for a surveys in nonlocal diffusion.

In particular, if we incorporate nonlocal diffusion into model (1.3), then

\[
\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} J(x-y)[u(y,t) - u(x,t)]dy - au(x,t) + bu(x, t-\tau)[1 - u(x, t - \tau)],
\]

(1.5)

in which the parameters are same to that in (1.1) and (1.4).

By the background of model (1.5) and the traveling wave front, we are interested in the traveling wave fronts of (1.5) which connects the trivial equilibrium state 0 with the positive equilibrium state. Using the abstract result in [10], we establish the existence of traveling wave fronts which admit asymptotic behavior like an exponential function when traveling wave coordinate goes to negative infinity. Our main result implies that there is a moving zone of transition from the disease-free state to the infective state [8, 11]. We also should note that precisely asymptotic behavior of traveling wave fronts is very important in further investigation of the property of traveling wave fronts, such as the uniqueness of traveling wave fronts in nonlocal diffusion equation [3]. Based on the result in this paper, we shall further investigate the properties of the traveling wave fronts in our forthcoming papers.
2 Preliminaries

Before investigating the traveling wave fronts of (1.5), we first consider the following model

\[
\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} J(x-y)[u(y,t) - u(x,t)]dy + f(u(x,t), u(x,t-\tau)),
\]

(2.1)

where all the parameters are same to that in Section 1. For model (2.1), its definition of traveling wave fronts is given as follows.

**Definition 2.1** A traveling wave solution of (2.1) is a special solution with form \( u(x,t) = \phi(x + ct) \) in which \( c > 0 \) is the wave speed by which the wave profile \( \phi \in C^1(\mathbb{R}, \mathbb{R}) \) propagates through \( \mathbb{R} \). Moreover, if \( \phi(t) \) is monotone in \( t \in \mathbb{R} \), then it is called a traveling wave front.

By Definition 2.1, the traveling wave fronts \( \phi(x + ct) \) of (2.1) must satisfy the following functional differential equation

\[
c\phi'(t) = \int_{-\infty}^{\infty} J(t-y)[\phi(y) - \phi(t)]dy + f(\phi(t), \phi(t - c\tau)), t \in \mathbb{R}.
\]

(2.2)

Motivated by the practical background of the traveling wave fronts in material science [1], physics [2], we also require that \( \phi \) satisfies the following asymptotic boundary condition

\[
\lim_{t \to -\infty} \phi(t) = 0, \quad \lim_{t \to \infty} \phi(t) = K
\]

(2.3)

with \( K > 0 \) and \( f(0,0) = f(K,K) = 0 \).

In order to apply the results in Pan et al. [10], we give some conditions on \( f \) and \( J \) as follows

**(f1)** For any \( u, v \in [0, K] \), \( f(u,v) \) is nondecreasing in \( v \);

**(f2)** For any \( u, v \in [0, K] \), \( f(u,v) \) is Lipschitz continuous in \( u, v \);

**(J1)** \( \int_{\mathbb{R}} J(x)u(x)dx \geq 0 \) for all bounded \( u(x) \geq 0 \);

**(J2)** For any \( \lambda \geq 0 \), \( 0 < \int_{\mathbb{R}} J(x)e^{\lambda x}dx < \infty \);

**(J3)** \( \int_{\mathbb{R}} J(x)u(x)dx = \int_{\mathbb{R}} J(-x)u(x)dx \) for any bounded \( u(x), x \in \mathbb{R} \);

**(J4)** \( \int_{\mathbb{R}\setminus0} J(x)dx > 0 \).

The definition of upper and lower solutions of (2.2) are given as follows.
Definition 2.2 A continuous function $\phi(t), t \in \mathbb{R}$ is called an upper (lower) solution of (2.2), if $\phi'(t), t \in \mathbb{R} \setminus \mathbb{T}$ exists and is bounded, moreover $\phi', \phi$ also satisfy the following inequality

$$c\phi'(t) \geq (\leq) \int_{-\infty}^{\infty} J(t-y)[\phi(y) - \phi(t)]dy + f(\phi(t), \phi(t - c\tau)), t \in \mathbb{R} \setminus \mathbb{T},$$

where $\mathbb{T}$ contains only finite point of $\mathbb{R}$.

Lemma 2.3 [10] Assume that $(f1) - (f2)$ and $(J1) - (J2)$ hold. If (2.2) has a pair of upper and lower solutions $\overline{\phi}(t), \underline{\phi}(t)$ such that

(a) $\sup_{s \leq t} \underline{\phi}(s) \leq \overline{\phi}(t)$ for $t \in \mathbb{R}$;

(b) $f(u, u) \neq 0, u \in (0, \inf_{t \in \mathbb{R}} \underline{\phi}(t)] \cup [\sup_{t \in \mathbb{R}} \overline{\phi}(t), K)$.

Then (2.2)-(2.3) have a monotone solution.

3 Main Result and the Proof

In this section, we shall prove the existence of traveling wave fronts of (1.5) by Lemma 2.3. Let $u(x, t) = \phi(x + ct)$ be the traveling wave fronts of (1.5), then $\phi$ satisfies

$$c\phi'(t) = \int_{-\infty}^{\infty} J(t-y)[\phi(y) - \phi(t)]dy - a\phi(t) + b\phi(t - c\tau)[1 - \phi(t - c\tau)], t \in \mathbb{R},$$

and we are interested in the following asymptotic boundary conditions

$$\lim_{t \to -\infty} \phi(t) = 0, \lim_{t \to \infty} \phi(t) = \frac{b - a}{b}$$

provided that $b > a > 0$.

For $0 \leq u, v \leq \frac{b-a}{b}$ with $b > a > 0$, define $g(u, v)$ as follows

$$g(u, v) = -au + bv(1 - v).$$

Lemma 3.1 Assume that $2a \geq b > a > 0$. Then $g(u, v)$ satisfies $(f1) - (f2)$.

Lemma 3.1 is clear, and we omit its proof here. By Lemmas 2.3 and 3.1, (3.1) and (3.2) have a monotone solution if they have a pair of proper upper and lower solutions. By the assumptions (J1)-(J4), define equation

$$\Delta(\lambda, c) = \int_{-\infty}^{\infty} J(y) \left[ e^{\lambda y} - 1 \right] dy - c\lambda - a + be^{-\lambda c\tau}$$

for $\lambda \geq 0$. It is clear that (3.3) is well defined and satisfies the follows.
Lemma 3.2 There exists $c^* > 0$ such that (3.3) has two different positive real roots $\lambda_1(c) < \lambda_2(c)$ if $c > c^*$ while (3.3) has no real root if $c < c^*$. Moreover,

$$\Delta(\lambda, c) = \begin{cases} > 0, & \lambda \in (0, \lambda_1(c)) \cup (\lambda_2(c), \infty), \\ < 0, & \lambda \in (\lambda_1(c), \lambda_2(c)). \end{cases}$$

By the constants in Lemma 3.2, define continuous functions as follows

$$\overline{\varphi}(t) = \min \left\{ e^{\lambda_1(c)t}, \frac{b - a}{b} \right\}, \quad \underline{\varphi}(t) = \max \left\{ 0, e^{\lambda_1(c)t} - qe^{\eta_1(c)t} \right\}$$

with $\eta \in (1, \min \{2, \frac{\lambda_2(c)}{\lambda_1(c)}\})$ and $q > 1$.

Lemma 3.3 $\overline{\varphi}(t)$ is an upper solution of (3.1).

Proof. It suffices to prove that $\overline{\varphi}(t)$ satisfies the definition of upper solutions. If $\overline{\varphi}(t) = e^{\lambda_1(c)t}$, then $\overline{\varphi}(y) \leq e^{\lambda_1(c)y}$ and

$$\int_{-\infty}^{\infty} J(t - y)[\overline{\varphi}(y) - \overline{\varphi}(t)]dy - a\overline{\varphi}(t) + b\overline{\varphi}(t - c\tau)[1 - \overline{\varphi}(t - c\tau)] - c\overline{\varphi}'(t)$$

$$\leq \int_{-\infty}^{\infty} J(t - y)[e^{\lambda_1(c)y} - e^{\lambda_1(c)t}]dy - ae^{\lambda_1(c)t} + be^{\lambda_1(c)(t-c\tau)} - c\lambda_1(c)e^{\lambda_1(c)t}$$

$$= e^{\lambda_1(c)t} \Delta(\lambda_1(c), c) = 0.$$

If $\overline{\varphi}(t) = 1$, then the result is clear. The proof is complete. \(\square\)

Lemma 3.4 If $q > 1 - \frac{b}{\Delta(\eta_1(c), c)}$, then $\overline{\varphi}(t)$ is a lower solution of (3.1).

Proof. It is sufficient to prove that $\underline{\varphi}(t)$ satisfies the definition of lower solution, and the result is clear if $\underline{\varphi}(t) = 0$. If $\underline{\varphi}(t) = e^{\lambda_1(c)t} - qe^{\eta_1(c)t}$, then it is easy to see that $t < 0$ and

$$\int_{-\infty}^{\infty} J(t - y)[\underline{\varphi}(y) - \underline{\varphi}(t)]dy - a\underline{\varphi}(t) + b\underline{\varphi}(t - c\tau)[1 - \underline{\varphi}(t - c\tau)] - c\overline{\varphi}'(t)$$

$$\leq \int_{-\infty}^{\infty} J(t - y)[e^{\lambda_1(c)y} - qe^{\eta_1(c)y} - (e^{\lambda_1(c)t} - qe^{\eta_1(c)t})]dy - a(e^{\lambda_1(c)t} - qe^{\eta_1(c)t})$$

$$+ b(e^{\lambda_1(c)(t-c\tau)} - qe^{\eta_1(c)(t-c\tau)}) - be^{2\lambda_1(c)(t-c\tau)} - c(e^{\lambda_1(c)t} - qe^{\eta_1(c)t})$$

$$= -q\Delta(\eta_1(1), c)e^{\eta_1(1)c}t - be^{2\lambda_1(c)(t-c\tau)} > 0$$

since $q > 1 - \frac{b}{\Delta(\eta_1(1), c)}$ holds. The proof is complete. \(\square\)

What we have done implies the following main result of this paper.

Theorem 3.5 Assume that $2a \geq b > a > 0$ and $c > c^*$ hold and the assumptions (J1)-(J4) are true. Then (3.1) and (3.2) has a monotone solution $\phi(t)$ such that $\lim_{t \to -\infty} \phi(t)e^{-\lambda_1(c)t} = 1$.
References


Received: March 8, 2008