Continuity and Boundedness of Superposition Operators on $W_{φ,X}[a,∞)$

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Abstract

For any linear operator on a Banach space, the continuity and the boundedness are equivalent. It may fail for non-linear operators. In this paper, we concern about one of the very useful non-linear operators that is it so-called the superposition operator. The main goal is to construct the sufficient and necessary conditions for the continuity as well for the boundedness of the operator.

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1 Introduction

In recent years, function spaces and operator theory has received a lot of attention from mathematicians in the areas of modern analysis and applied mathematics. As we have known that for any linear operator on a Banach space, the continuity and the boundedness are equivalent. It may fail for non-linear operators. Following the facts, we intend to discuss some recent achievements in the operator theory. We focus the discussion on the very useful non-linear operator that is it so-called the superposition operator. The key references are [1],[4],[5], and [6]. In this introductory section we describe some basic notions.

As usual, the real and natural numbers systems will be denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively.

Let $X$ be a Banach lattice and $(E, \Sigma, \mu)$, where $E \subset \mathbb{R}$, a measure space. A function $s : E \to X$ is called a simple function if there exist a collection of disjoint measurable sets $E_1, E_2, \ldots, E_n \subset E$ with $\bigcup_{k=1}^n E_k = E$ and a
sequence of $a_1, a_2, \ldots, a_n$ in $X$ such that

$$s(x) = \sum_{k=1}^{n} a_k \chi_{A_k}(x), \quad x \in E$$  \hspace{1cm} (1)

If $s$ is a simple function on $E$, then its representation is not unique. If each $a_k$'s in the expression (1) are distinct and nonzero then (1) is called a canonical representation of $s$.

A function $f : E \to X$ is said to be measurable if there exists a sequence of simple functions $\{s_n\}$ from $E$ into $X$ such that

$$\lim_{n \to \infty} \|f(x) - s_n(x)\| = 0$$

almost everywhere on $E$.

Let $s : E \to X$ be a simple function with the canonical representation is given as (1). For any measurable set $A \subset E$, then the vector

$$\int_A s(x) d\mu(x) = \sum_{k=1}^{n} a_k \mu(A \cap A_k),$$

if it is exists, is uniquely determined. The vector $\int_A s(x) d\mu(x)$ is then called the integral of $s$ on $A$.

The measurable function $f : E \to X$ is said to be integrable on a measurable set $A \subset E$ if there exists a sequence of simple functions $\{s_n\}$ from $E$ into $X$ such that $\lim_{n \to \infty} \|f(x) - s_n(x)\| = 0$ almost everywhere on $E$ and

$$\lim_{n \to \infty} \int_A \|f(x) - s_n(x)\| d\mu(x) = 0$$

Further, the integral of $f$ on $A$ is given by

$$\int_A f(x) d\mu(x) = \lim_{n \to \infty} \int_A s_n(x) d\mu(x)$$  \hspace{1cm} (2)

Let $(E, \Sigma, \mu)$, where $E \subset \mathbb{R}$, be a measure space, $A \in \Sigma$, and $X$ a Banach lattice. The collection of all measurable $X$-valued functions on $A$ will be denoted by $\mathcal{M}_X(A)$. We then define

$$L_1(A) = \{ f \in \mathcal{M}_\mathbb{R}(A) : \int_A |f(x)| d\mu < \infty \},$$

$$L_1(A) = \{ f \in \mathcal{M}_X(A) : \int_A |f(x)| d\mu = \alpha, \text{ for some } \alpha \in X \}.$$  \hspace{1cm} (2)

For any $f \in \mathcal{M}_X(A)$ and $B \subset A$, we define the function $f_B$ as

$$f_B(t) = \begin{cases} f(t) & , t \in B \\ 0 & , t \notin B \end{cases}$$

It is clear that $f_B \in \mathcal{M}_X(A)$ for every $f \in \mathcal{M}_X(A)$ and measurable set $B \subset A$. Particularly, for any (fixed) $a \in \mathbb{R}$, we write $f_x$ instead of $f_{[a,x]}$. 
Definition 1.1 Let $H$ be a linear space over the real number system $\mathbb{R}$. The non-negative function $\| \cdot \| : H \to [0, \infty)$ is called an $F$-norm if

i. $\| f \| = 0 \iff f = 0,$

ii. $\|- f\|=\| f\|$ for every $f \in H,$

iii. $\|f + g\| \leq \|f\| + \|g\|$ for every $f, g \in H,$

iv. If the sequence $\{a_n\} \subset \mathbb{R}$ is convergent to some $a \in \mathbb{R}$ and $\{f^{(n)}\} \subset H$ is a sequence such that $\{|f^{(n)} - f|\}$ converges to 0 for some $f \in H$ then $\{|a_n f^{(n)} - af|\}$ converges to 0.

Further, the linear space $H$ equipped with the $F$-norm $\| \cdot \|$, denoted by $(H, \| \cdot \|)$, is called an $F$-normed space. If the $F$-norm have determined, we write shortly the $F$-normed space by $H$. A complete $F$-normed space is called a Fréchet space or an $F$-space.

An $F$-space $H \subset \mathcal{M}_X(A)$ is called an $FK$-space if the canonical map $p_x : H \to X,$

$$p_x(f) = f(x),$$

is continuous for every $x \in A$. An $F$-space $H \subset \mathcal{M}_X(A)$ is called an $AK$-space if $f_B, \alpha X_B \in H$ for every $f \in H$, $\alpha \in X$, and measurable set $B \subset A$ with $\mu(B) \leq \infty$, and $\lim_{\mu(B) \to \mu(A)} \|f_B - f\| = 0$.

As usual, for any Banach lattice $X$, $X^+$ denotes the collection of all positive vectors in $X$, namely $X^+ = \{ x \in X; x \geq 0 \}$. If $X$ and $Y$ are Banach lattices, respectively, then a function $\phi : X \to Y$ is called an $N$-function if it holds the followings

i. $\phi(x) = 0 \iff x = 0,$

ii. $\phi$ is increasing on $X^+$,

iii. $\phi$ is continuous on $X$, and

iv. $\phi(-x) = \phi(x)$ for every $x \in X$.

The $N$-function $\phi$ is said to satisfy the $\Delta_2$-condition if there exists a real number $M > 0$ such that $\phi(2x) \leq M\phi(x)$ for every $x \in X^+$.

Let $X$ be a Banach lattice and $\phi$ $N$-function that satisfies the $\Delta_2$-condition. For any real number $a > 1$, we define

$$W_{\phi,X}[a, \infty) = \{ f \in \mathcal{M}_X[a, \infty) : \lim_{x \to \infty} \rho_x(f) = 0 \},$$

where

$$\rho_x(f) = \frac{1}{x} \| \int_a^x \phi(f(t))d\mu \|.$$
We can show that the functions $\rho$ and $\| \cdot \|$, 
\[
\rho(f) = \sup \{ \rho_x(f) : x \in [a, \infty) \},
\]
\[
\|f\| = \inf \{ \epsilon > 0 : \rho(\frac{f}{\epsilon}) \leq \epsilon \},
\]
are a modular and an $F$-norm on $W_\phi X[a, \infty)$, respectively. Further, $W_\phi X[a, \infty)$ is a solid, $FK$- and $AK$-space. We also observe the following lemmas.

**Lemma 1.2** Let $f \in W_\phi X[a, \infty)$, then for every real number $\beta > 0$ there exists an $\alpha > 0$ such that the condition $\|f\| < \alpha$ implies $\rho(f) < \beta$.

**Lemma 1.3** Let $f \in W_\phi X[a, \infty)$, then for every two real numbers $\alpha, \gamma > 0$ there exists a real number $\beta > 0$ such that $\|f\| \leq \alpha$ whenever $\rho(\gamma f) \leq \beta$.

## 2 Superposition Operators

Let $X$ and $Y$ be Archimedean, $\sigma$-order complete Banach lattices, respectively and $g(., .): A \times X \to Y$ a function such that $g(., t)$ is measurable for any $t \in X$ and $g(x, 0) = 0$ for each $x \in A$. The operator $P_g : \mathcal{M}_X(A) \to \mathcal{M}_Y(A)$, 
\[
P_g(f)(x) = g(x, f(x)), \quad f \in \mathcal{M}_X(A), \quad x \in A
\]
is called the superposition operator.

Through out this paper we always assume that the Banach lattices $X$ and $Y$ are Archimedean and $\sigma$-order complete, unless otherwise stated. We observe the following lemma.

**Lemma 2.1** Let $A \in \Sigma$, $\mathcal{D} \subset \mathcal{M}_Y(A)$ be an $FK$- and $AK$-space and the function $g(., .): A \times X \to Y$ satisfies $g(., t)$ is measurable on $A$ for every $t \in X$, $g(x, 0) = 0$, and $g(x, .)$ is continuous on $X$ for every $x \in A$. If there exist $\alpha \in Y^+$ and $\beta \in Y^+ - \{0\}$ such that the condition $\frac{1}{\mu(B)} \int_B \phi(f(x))d\mu \leq \beta$ implies $\int_B |g(x, f(x))|d\mu \leq \alpha$ for every $f \in \mathcal{D}$ and measurable set $B \subset A$ with $\mu(B) \neq 0$, then for every measurable set $B \subset A$ there exists a non-negative function $h \in \mathcal{L}_1(B) \subset \mathcal{M}_Y(B)$ with $\int_B h(x)d\mu \leq \alpha$ such that for any $x \in B$, 
\[
|g(x, t)| \leq h(x) + 2\alpha \beta^{-1} \mu(B)^{-1} \phi(t),
\]
as $\frac{\phi(t)}{\mu(B)} \leq \beta$.

**Proof:** Let $B \subset A$ be a measurable set and $t \in X$. We define the function $k$ on $A \times X$ and the function $h$ on $B$ as follow
\[
k(x, t) = \begin{cases} 
|g(x, t)| - 2\alpha \beta^{-1} \mu(B)^{-1} \phi(t), & \text{whenever } |g(x, t)| \geq 2\alpha \beta^{-1} \mu(B)^{-1} \phi(t) \\
0, & \text{otherwise}
\end{cases}
\]
Continuity and boundedness of superposition operators

\[ h(x) = \sup\{k(x,t) : \frac{\phi(t)}{\mu(B)} \leq \beta\} = k(x,u(x)) \]

It is clear that the both functions \( k \) and \( h \) depend on \( B \). Since \( \phi \) is continuous, then the \( h(x) \) is uniquely determined. From the definition above, it is clear that the function \( h \) is non-negative on \( B \). Further, since \( g(x,:) \) and \( \phi \) are both continuous on \( X \) then \( k(x,:) \) is continuous on \( X \) for every \( x \in B \). The continuity of \( \phi \) then implies the set \( \{t \in X : \frac{\phi(t)}{\mu(B)} \leq \beta\} \) is closed and bounded, and hence \( h \) is bounded on \( B \). What remain to show is \( h \in L_1(B) \).

Let \( B \subset A \) be any measurable set, then \( \int_B \phi(u(x))d\mu \) can be decomposed as

\[ \int_B \phi(u(x))d\mu = \int_{A_1} \phi(u(x))d\mu + \int_{A_2} \phi(u(x))d\mu + \ldots + \int_{A_n} \phi(u(x))d\mu, \]

where \( A_1, A_2, \ldots, A_n \subset B \) are measurable, \( \cup_{i=1}^n A_i = B \), \( \mu(A_i \cap A_j) = 0 \), for \( i \neq j \), \( i, j = 1,2,\ldots,n \), and \( \beta \leq \frac{1}{n} \int_{A_i} \phi(u(x))d\mu \leq \beta \), for \( i = 1,2,\ldots,n-1 \), and \( 0 \leq \frac{1}{n} \int_{A_n} \phi(u(x))d\mu \leq \beta \). Hence,

\[
\int_B h(x)d\mu = \int_B k(x,u(x))d\mu \leq \int_B |g(x,u(x))|d\mu - 2\alpha \beta^{-1} \mu(B)^{-1} \int_B \phi(u(x))d\mu |
\]

\[
= \sum_{i=1}^n \int_{A_i} |g(x,u(x))|d\mu - 2\alpha \beta^{-1} \mu(B)^{-1} \sum_{i=1}^n \| \int_{A_i} \phi(u(x))d\mu \|
\]

\[
\leq n \alpha - 2\alpha \beta^{-1} \mu(B)^{-1} \sum_{i=1}^n \| \int_{A_i} \phi(u(x))d\mu \| = \alpha.
\]

Thus, \( h \in L_1(B) \). Further, by the definition of \( h \) and \( k \), the assertion follows.

\[ \square \]

Notice that the real numbers system \( \mathbb{R} \) is a Banach lattice. Therefore, following the Lemma 2.1 we have the corollary below.

**Corollary 2.2** Let \( A \subset \mathbb{R} \) be a measurable set, \( D \subset \mathcal{M}_\mathbb{R}(A) \) an FK- and AK-space, and \( g(.,.) : A \times X \rightarrow \mathbb{R} \) a function such that \( g(.,t) \) measurable for every \( t \in X \), \( g(x,0) = 0 \) and \( g(x,:) \) is continuous on \( X \) for every \( x \in A \). If there are real numbers \( \alpha, \beta > 0 \) such that for any \( f \in D \) and a measurable set \( B \subset A \) with \( \mu(B) \neq 0 \), the condition \( \frac{1}{\mu(B)} \int_B \phi(f(x))d\mu \leq \beta \) implies \( \int_B |g(x,f(x))|d\mu \leq \alpha \), then for any measurable set \( B \subset A \) there exists a non negative function \( h \in L_1(B) \) with \( \int_B h(x)d\mu \leq \alpha \) such that for every \( x \in B \),

\[ |g(x,t)| \leq h(x) + 2\alpha \beta^{-1} \mu(B)^{-1} \phi(t), \]

whenever \( \frac{\phi(t)}{\mu(B)} \leq \beta \).

If \( g \) satisfies a certain condition, then we can show that \( P_g : W_{\phi,X}[a,\infty) \rightarrow \mathcal{L}_1[a,\infty) \subset \mathcal{M}_Y[a,\infty) \).
**Theorem 2.3** Let \( g(.,.) : [a, \infty) \times X \to Y \) be a function such that \( g(.,t) \) is measurable for any \( t \in X \), \( g(x,0) = 0 \), dan \( g(x,.) \) is continuous on \( X \) for every \( x \in [a, \infty) \). If there exist \( \alpha \in Y^+ \), \( \beta \in Y^+ - \{0\} \), and for any \( s \in [a, \infty) \) there exists a non-negative function \( h \in L_1[a, s] \) with \( \int_a^s h(x)d\mu \leq \alpha \) such that for any \( x \in [a, s] \),

\[
|g(x,t)| \leq h(x) + 2\alpha\|\beta^{-1}\|s^{-1}\phi(t),
\]
as \( \frac{\phi(t)}{s} \leq \beta \), then the superposition operator \( P_g \) maps \( W_{\phi,X}[a, \infty) \) into \( L_1[a, \infty) \).

**Proof:** Let \( f \in W_{\phi,X}[a, \infty) \), then there exists a real number \( M > 0 \) such that for every \( s \geq M \),

\[
\frac{1}{s}\int_a^s \phi(f(x))d\mu < \beta.
\]

Following the hypothesis, for every \( s \geq M \) we have

\[
\int_a^s |g(x,f(x))|d\mu \leq \int_a^s h(x)d\mu + 2\alpha\|\beta^{-1}\|s^{-1}\|\int_a^s \phi(f(x))d\mu\|
\]

\[
\leq \alpha + 2\alpha\|\beta^{-1}\||\beta\| = 3\alpha.
\]

Since it holds for every \( s \geq M \) then \( \int_a^\infty |g(x,f(x))|d\mu \) exists, or \( g(.,f(.)) \in L_1[a, \infty) \). Thus, \( P_g : W_{\phi,X}[a, \infty) \to L_1[a, \infty) \subset M_Y[a, \infty) \). \( \square \)

The Theorem 2.3 states the sufficient condition so that the superposition operator \( P_g \) maps the space \( W_{\phi,X}[a, \infty) \) onto \( L_1[a, \infty) \subset M_Y[a, \infty) \). It is very difficult to prove the converse of the theorem. However, if the function \( g \) is real valued, i.e. \( g(.,.) : [a, \infty) \times X \to \mathbb{R} \), we can construct the sufficient and necessary conditions so that \( P_g \) maps \( W_{\phi,X}[a, \infty) \) onto \( L_1[a, \infty) \).

**Theorem 2.4** Let \( g(.,.) : [a, \infty) \times X \to \mathbb{R} \) be a function such that \( g(.,t) \) is measurable for every \( t \in X \), \( g(x,0) = 0 \), and \( g(x,.) \) is continuous on \( X \) for every \( x \in [a, \infty) \). The superposition operator \( P_g \) maps the space \( W_{\phi,X}[a, \infty) \) onto \( L_1[a, \infty) \) if and only if there exist real numbers \( \alpha, \beta > 0 \) and for each \( s \in [a, \infty) \) there exists a non negative function \( h \in L_1[a, s] \) satisfies \( \int_a^s h(x)d\mu \leq \alpha \) such that for every \( x \in [a, s] \),

\[
|g(x,t)| \leq h(x) + 2\alpha\beta^{-1}s^{-1}\phi(t),
\]

whenever \( \frac{\phi(t)}{s} \leq \beta \).

**Proof:** The sufficiency follows from the Theorem 2.3 by taking \( Y = \mathbb{R} \).

For the necessity, the functional \( F : W_{\phi,X}[a, \infty) \to \mathbb{R} \),

\[
F(f) = \int_a^\infty |g(x,f(x))|d\mu,
\]
is continuous and orthogonally additive (See the reference [6]). So, there exists a real number \( \eta > 0 \) such that for every \( f \in W_{\phi,X}[a, \infty) \) with \( \|f\| \leq \eta \) we have

\[
\int_a^\infty |g(x, f(x))|d\mu < 1.
\]

It is easy to prove that there exists a real number \( \beta > 0 \) such that the condition \( \rho(f) < \beta \) implies \( \|f\| < \eta \). Thus, there exists \( \beta > 0 \) such that for every \( f \in W_{\phi,X}[a, \infty) \),

\[
\int_a^\infty |g(x, f(x))|d\mu < 1,
\]

whenever \( \rho(f) < \beta \). These implies for every \( s \in [a, \infty) \),

\[
\int_a^s |g(x, f(x))|d\mu < 1,
\]

whenever \( \frac{1}{s} \int_a^s \phi(f(x))d\mu \leq \beta \).

Further, if we take \( \alpha = \sup \{ \int_a^s |g(x, f(x))|d\mu : \frac{1}{s} \int_a^s \phi(f(x))d\mu \leq \beta \} \), then the Theorem 2.3 implies the existence of the non negative function \( h \in L_1[a, s] \) satisfies \( \int_a^s h(x)d\mu \leq \alpha \), such that for any \( x \in [a, s] \),

\[
|g(x, t)| \leq h(x) + 2\alpha \beta^{-1}s^{-1}\phi(t),
\]

as \( \frac{\phi(t)}{s} \leq \beta \). \( \Box \)

Following the Theorem 2.4, for the sequel we always assume that the generator of the superposition operator \( P_g \) is the function \( g(.,.) : [a, \infty) \times X \rightarrow \mathbb{R} \), unless otherwise stated.

3 Continuity of Superposition Operators on \( W_{\phi,X}[a, \infty) \)

Let \( \mathcal{D} \subset \mathcal{M}_X[a, \infty) \) be a space modulared by \( \rho \). The superposition operator \( P_g : \mathcal{D} \rightarrow L_1[a, \infty) \) is said to be continuous at \( f \in \mathcal{D} \) if for every real number \( \epsilon > 0 \) there exists a real number \( \delta > 0 \) such that for every \( h \in \mathcal{D} \) with \( \|f - h\| < \delta \) we have

\[
\|P_g(f) - P_g(h)\| < \epsilon.
\]

The operator \( P_g \) is said to be modular continuous (\( \rho \)-continuous) at \( f \in \mathcal{D} \) if for every pair of real numbers \( \alpha, \epsilon > 0 \) there exists a real number \( \delta > 0 \) such that for each \( h \in \mathcal{D} \)

\[
\|P_g(f) - P_g(h)\| < \epsilon,
\]

whenever \( \rho(\alpha(f - h)) < \delta \). Further, \( P_g \) is said to be continuous (\( \rho \)-continuous) on \( \mathcal{D} \) if \( P_g \) is continuous (\( \rho \)-continuous) at each \( f \in \mathcal{D} \).
Theorem 3.1 The superposition operator \( P_g : W_{\phi,X}[a, \infty) \to L_1[a, \infty) \) is \( \rho \)-continuous if and only if \( P_g \) is continuous.

Proof: For the sufficiency, take any \( f \in W_{\phi,X}[a, \infty) \) and any real number \( \epsilon > 0 \). Since \( P_g \) is \( \rho \)-continuous, then there exists a real number \( \beta > 0 \) such that for any \( h \in W_{\phi,X}[a, \infty) \) with \( \rho(f - h) < \beta \) we have

\[
\|P_g(f) - P_g(h)\| < \epsilon.
\]

Following the Lemma 1.2, there exists \( \alpha > 0 \) such that

\[
\rho(f - h) < \beta,
\]

whenever \( \|f - h\| < \alpha \).

For the necessity, let \( f \in W_{\phi,X}[a, \infty) \) and \( \gamma, \epsilon > 0 \) be arbitrarily. Since \( P_g \) is continuous at \( f \), then there exists a real number \( \alpha > 0 \) such that for any \( h \in W_{\phi,X}[a, \infty) \) with \( \|f - h\| < \alpha \),

\[
\|P_g(f) - P_g(h)\| < \epsilon.
\]

By the Lemma 1.3, for the \( \alpha, \gamma > 0 \) there exists \( \beta > 0 \) such that

\[
\|f - h\| < \alpha,
\]

whenever \( \rho(\gamma(f - h)) < \beta \). These complete the proof. \( \square \)

Theorem 3.2 The operator \( P_g : W_{\phi,X}[a, \infty) \to L_1[a, \infty) \) is \( \rho \)-continuous if and only if the function \( g(x, \cdot) \) is continuous on \( X \) for every \( x \in [a, \infty) \).

Proof: The sufficiency: Let \( \gamma > 0 \) be any real number, then there exists a positive integer \( n_0 \) such that \( \frac{1}{n_0} < \gamma \). Since the function \( g(x, \cdot) \) is continuous on \( X \) for every \( x \in [a, \infty) \) and \( P_g : W_{\phi,X}[a, \infty) \to L_1[a, \infty) \), then by the Theorem 2.4 there exist real numbers \( \alpha, \beta > 0 \) and for every \( s \in [a, \infty) \) there exists a non negative function \( h \in L_1[a, s] \) with \( \int_a^s h(x)d\mu \leq \alpha \) such that for each \( x \in [a, s] \),

\[
|g(x,t)| \leq h(x) + 2\alpha\beta^{-1}s^{-1}\phi(\frac{1}{2n_0}t),
\]

whenever \( \frac{\phi(\frac{1}{n_0}s)}{s} \leq \beta \). Let \( f \in W_{\phi,X}[a, \infty) \), i.e.

\[
\lim_{s \to \infty} \rho_s(\frac{1}{n_0}f) = 0,
\]

then for every real number \( \epsilon > 0 \) there is an \( M > a \) such that for any \( s \geq M \),

\[
\rho_s(\frac{1}{2n_0}f) \leq \rho_s(\frac{1}{n_0}f) < \frac{\beta\epsilon}{16\alpha}.
\]
Then, by choosing \( k \in W_{\phi,X}[a, \infty) \) such that \( \rho(\frac{1}{n_0}(f - k)) < \frac{\beta \epsilon}{16\alpha} \), then for any \( s \geq M \) we have

\[
\rho_s\left(\frac{1}{2n_0}k\right) = \rho_s\left(\frac{1}{2n_0}(k - f) + \frac{1}{2n_0}f\right) \\
\leq \rho_s\left(\frac{1}{n_0}(k - f)\right) + \rho_s\left(\frac{1}{n_0}f\right) \\
\leq \frac{\beta \epsilon}{16\alpha} + \frac{\beta \epsilon}{16\alpha} = \frac{\beta \epsilon}{8\alpha}.
\]

Since the function \( g(x,.) \) is continuous on \( X \) for every \( x \in [a, \infty) \), then there exists \( \delta > 0 \), \( \delta < \frac{\beta \epsilon}{16\alpha} \), such that for any \( x \in [a, s] \), the condition \( \|f(x) - k(x)\| < n_0\delta \) implies

\[
\int_a^s |g(x, f(x)) - g(x, k(x))|d\mu < \frac{\epsilon}{4}.
\]

By taking \( b \in [a, s] \) such that \( \int_b^s h(x)d\mu < \frac{\epsilon}{8} \), then for each \( s \geq M \) we have

\[
\int_a^s |g(x, f(x)) - g(x, k(x))|d\mu \leq \int_a^b |g(x, f(x)) - g(x, k(x))|d\mu \\
+ \int_b^s |g(x, f(x))|d\mu + \int_b^s |g(x, k(x))|d\mu \\
< \frac{\epsilon}{4} + 2 \int_b^s h(x)d\mu \\
+ 2\alpha\beta^{-1}s^{-1} \int_b^s \phi(\frac{1}{2n_0}f(x))d\mu \\
+ 2\alpha\beta^{-1}s^{-1} \int_b^s \phi(\frac{1}{2n_0}k(x))d\mu \\
< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\]

Thus, for every \( f \in W_{\phi,X}[a, \infty) \) and real numbers \( \epsilon, \gamma > 0 \) there exists \( \delta > 0 \) such that for any \( k \in W_{\phi,X}[a, \infty) \) with \( \rho(\gamma(f - k)) < \delta \), which implies \( \rho(\frac{1}{n_0}(f - k)) < \frac{\beta \epsilon}{16\alpha} \), we have

\[
\|P_g(f) - P_g(k)\| < \epsilon,
\]

i.e. \( P_g \) is \( \rho \)-continuous at \( f \).

The necessity: Let \( x \in [a, \infty) \) and \( t \in X \). We are going to prove that \( g(x,.) \) is continuous at \( t \).

Notice that, for any measurable set \( E \subset [a, \infty) \), \( t \chi_E \in W_{\phi,X}[a, \infty) \). By hypothesis, then for any real number \( \epsilon > 0 \) there is \( \delta_1 > 0 \) such that for any \( f \in W_{\phi,X}[a, \infty) \) with \( \rho(f - t \chi_E) < \delta_1 \), we have

\[
\|P_g(f) - P_g(t \chi_E)\| < \epsilon.
\]
Further, take any \( p \in X \), then for any (fixed) \( s \in [a, \infty) \),

\[
\rho_s(p \chi_E - t \chi_E) = \frac{1}{s} \int_a^s \phi(p \chi_E(x) - t \chi_E(x)) d\mu
\]

\[
= \frac{1}{s} \int_{E \cap [a,s]} \phi(p-t) d\mu = \frac{\phi(p-t)}{s} \mu(E \cap [a,s]).
\]

Since the function \( \phi \) is continuous on \( X \) then there is \( \delta > 0 \) such that the condition \( \| p-t \| < \delta \) implies

\[
\phi(p-t) < \frac{s\delta_1}{\mu(E \cap [a,s]) + 1},
\]

therefore

\[
\rho_s(p \chi_E - t \chi_E) = \frac{\phi(p-t)}{s} \mu(E \cap [a,s]) < \delta_1.
\]

These are followed by

\[
\| P_g(p \chi_E) - P_g(t \chi_E) \| < \epsilon,
\]

since the operator \( P_g \) is \( \rho \)-continuous. Thus, for any real number \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for every \( p \in X \) with \( \| p-t \| < \delta \) we have

\[
| g(x,p) - g(x,t) | < \epsilon,
\]

i.e. \( g(x,.) \) is continuous at \( t \). \( \Box \)

Following the Theorem 3.1 and Theorem 3.2, we then get the following corollary.

**Corollary 3.3** The superposition \( P_g : W_{\phi,X}[a, \infty) \rightarrow L_1[a, \infty) \) is continuous if and only if the function \( g(x,.) \) is continuous on \( X \) for every \( x \in [a, \infty) \).

### 4 Boundedness of Superposition Operators on \( W_{\phi,X}[a, \infty) \)

Let \( D \subset M_X[a, \infty) \) be a modulared spaces with the modular \( \rho \) and \( g(.,.) : [a, \infty) \times X \rightarrow \mathbb{R} \) a function such that \( g(.,t) \) is measurable for every \( t \in X \) and \( g(x,0) = 0 \) for every \( x \in [a, \infty) \). The superposition operator \( P_g : D \rightarrow L_1[a, \infty) \) is said to be **locally modular bounded** at \( f \in D \) if there exist real numbers \( \alpha, \beta, \gamma > 0 \) such that for every \( h \in D \) with \( \rho(\gamma(f-h)) \leq \alpha \) we have

\[
\| P_g(h) \| \leq \beta.
\]
Operator $P_g$ is said to be **locally bounded** at $f \in D$ if there exist real numbers $\alpha, \beta > 0$ such that for any $h \in D$ with $\|f - h\| \leq \alpha$ we have

$$\|P_g(h)\| \leq \beta.$$  

Further, operator $P_g$ is said to be **locally modular bounded** (locally bounded) if $P_g$ is locally modular bounded (locally bounded) at every $f \in D$. It is easy to prove that the superposition operator $P_g : D \to L_1[a, \infty)$ is locally modular bounded iff it is locally bounded.

**Theorem 4.1** Let $g(.,.) : [a, \infty) \times X \to \mathbb{R}$ be a function such that $g(.,t)$ is measurable for every $t \in X$, $g(x,0) = 0$ for every $x \in [a, \infty)$, and for every $s \in [a, \infty)$ and $n \in \mathcal{N}$ there exist non negative functions $f_1, h \in L_1[a, s]$ and there is a $\delta > 0$ such that for any $x \in [a, s]$,

$$|g(x,t)| \leq h(x) + x^{-1} f_1(x) \phi\left(\frac{1}{n}t\right),$$

as $\frac{1}{x^n} \phi\left(\frac{1}{n}t\right) \leq \delta$. The superposition operator $P_g : W_{\phi,X}[a, \infty) \to L_1[a, \infty)$ is locally modular bounded iff for every $x \in [a, \infty)$, the function $g(x,.)$ is bounded on any closed and bounded subset in $X$.

**Proof:** For the sufficiency, take any $x \in [a, \infty)$ and $t \in X$. If $E \subset [a, \infty)$ is any measurable set with $\mu(E) < \infty$ then $t \chi_E \in W_{\phi,X}[a, \infty)$. By the hypothesis there exists real numbers $\alpha_1, \beta, \gamma > 0$ such that for every $k \in X$, the condition $\rho(\gamma(t - k) \chi_E) \leq \alpha_1$ implies

$$\|P_g(k \chi_E)\| \leq \beta.$$  

Further, since $\phi$ is continuous on $X$, then there exists a real number $\alpha > 0$ such that

$$\rho(\gamma(t - k) \chi_E) \leq \alpha_1,$$

whenever $\|t - k\| \leq \alpha$. Thus, there exists a real number $\alpha, \beta > 0$ such that for any $x \in [a, \infty)$,

$$|g(x,k)| \leq \|P_g(k \chi_E)\| \leq \beta,$$

as $\|t - k\| \leq \alpha$. It means $g(x,.)$ is locally bounded at $t$. Further, if $A \subset X$ is closed and bounded then by the Heine-Borel Theorem, $g(x,.)$ is closed on $A$.

For the necessity, let $s \in [a, \infty)$ and $n \in \mathcal{N}$ be arbitrarily. By the hypothesis there exists a non negative functions $f_1, h \in L_1[a, s]$ and a real number $\delta > 0$ such that for any $x \in [a, s]$,

$$|g(x,t)| \leq h(x) + x^{-1} f_1(x) \phi\left(\frac{1}{n}t\right),$$
whenever $\frac{1}{x^n} \phi(\frac{1}{n} t) \leq \delta$. Further, suppose $f \in W_{\phi,X}[a, \infty)$, then there is a real number $M > a$ such that for every $x \geq M$ we have
\[
 \rho_x(\frac{1}{n} f) = \frac{1}{x} \int_a^x \phi(f(s)) d\mu \leq \frac{\delta}{4}.
\]
However, from the definition of the function $\phi$, we can choose $k \in W_{\phi,X}[a, \infty)$ such that
\[
 \frac{1}{x} \phi(\frac{1}{n_0} k) < \delta \quad \text{and} \quad \rho(\frac{1}{n_0} (f \ominus k)) < \frac{\delta}{4},
\]
for some $n_0 \in \mathcal{N}$. For any $x < M$, let
\[
m(x) = \sup \{|g(x,t)| : \phi(\frac{1}{n}(t \ominus f(x)) \leq \frac{\delta}{4}\}.
\]
Since $\phi$ is continuous on $X$ then the set
\[
 A = \{t : \phi(\frac{1}{n}(t \ominus f(x)) \leq \frac{\delta}{4}\}
\]
is closed and bounded in $X$. Therefore, by the hypothesis $m(x) < \infty$ for any $x < M$. Thus, for every $s > M$ and $k \in W_{\phi,X}[a, \infty)$ with $\rho(\frac{1}{n_0} (f \ominus k)) < \frac{\delta}{4}$ we have
\[
 \int_a^s |g(x,k(x))| d\mu = \int_a^M |g(x,k(x))| d\mu + \int_M^s |g(x,k(x))| d\mu \\
\leq \int_a^M m(x) d\mu + \int_M^s \{h(x) + x^{-1} f_1(x) \phi(\frac{1}{n_0} k(x))\} d\mu \\
\leq K + \|h\| + \frac{\delta}{2} \|f_1\|, \quad \text{for some } K > 0.
\]
If we take $s \to \infty$ then
\[
 \|P_g(k)\| \leq K + \|h\| + \frac{\delta}{2} \|f_1\|.
\]
Finally, by letting $\alpha = \frac{\delta}{4}$, $\beta = K + \|h\| + \frac{\delta}{2} \|f_1\|$, and $\gamma = \frac{1}{n_0}$ the assertion follows. \qed

**Corollary 4.2** Let $g(.,.): [a, \infty) \times X \to \mathbb{R}$ be a function as in Theorem 4.1. The superposition operator $P_g : W_{\phi,X}[a, \infty) \to L_1[a, \infty)$ is locally bounded iff for every $x \in [a, \infty)$, the function $g(x,.)$ is bounded on any closed and bounded subset in $X$.

Let $\mathcal{D} \subset \mathcal{M}_X[a, \infty)$ be a modulared space with the modular $\rho$ and $g(.,.): [a, \infty) \times X \to \mathbb{R}$ a function such that $g(.,t)$ is measurable for every $t \in X$
and \( g(x,0) = 0 \) for every \( x \in [a, \infty) \). The superposition operator \( P_g : \mathcal{D} \to L_1[a, \infty) \) is said to be modular bounded on \( \mathcal{D} \) if for every \( f \in \mathcal{D} \) and for any real constants \( \alpha, \beta > 0 \) there exists \( \delta > 0 \) such that for every \( h \in \mathcal{D} \),

\[
\|P_g(h)\| \leq \delta,
\]

whenever \( \rho(\alpha(f - h)) \leq \beta \). The operator \( P_g \) is said to be bounded on \( \mathcal{D} \) if for every real number \( \alpha > 0 \) there exists a \( \delta > 0 \) such that for every \( f \in \mathcal{D} \) with \( \|f\| \leq \alpha \), we have

\[
\|P_g(f)\| \leq \delta.
\]

We can easily prove the following theorems.

**Theorem 4.3** Let \( f \in W_{\phi,X}[a, \infty) \) and \( \alpha, \beta > 0 \), then there exits real numbers \( \alpha_1, \beta_1 > 0 \) such that for every \( h \in W_{\phi,X}[a, \infty) \) with \( \rho(\alpha(f - h)) \leq \beta \) we have \( \rho(\alpha_1 h) \leq \beta_1 \).

**Theorem 4.4** The superposition operator \( P_g : W_{\phi,X}[a, \infty) \to L_1[a, \infty) \) is modular bounded iff for every real numbers \( \alpha, \gamma > 0 \) there exists \( \beta > 0 \) such that for any \( h \in W_{\phi,X}[a, \infty) \) with \( \rho(\gamma h) \leq \alpha \) we have \( \|P_g(h)\| \leq \beta \).

**Theorem 4.5** The superposition operator \( P_g : W_{\phi,X}[a, \infty) \to L_1[a, \infty) \) is modular bounded iff it is bounded.

**Theorem 4.6** Let the function \( g(.,.) : [a, \infty) \times X \to \mathbb{R} \) satisfies the following conditions: \( g(.,t) \) is measurable for every \( t \in X \), \( g(x,0) = 0 \) for every \( x \in [a, \infty) \), and for every real number \( \beta > 0 \) and \( s \in [a, \infty) \) there exist non negative functions \( f, h \in L_1[a, s] \) such that for every \( x \in [a, s] \)

\[
|g(x, t)| \leq h(x) + \frac{1}{x} f(x) \phi(t),
\]

whenever \( \frac{\phi(t)}{x} \leq \beta \). If \( g(x,.) \) is bounded on \( X \) then \( P_g : W_{\phi,X}[a, \infty) \to L_1[a, \infty) \) is modular bounded.

**Proof:** Take any \( s \in [a, \infty) \) and any real number \( \beta > 0 \). By the hypothesis there exist non negative functions \( f, h \in L_1[a, s] \) such that for every \( x \in [a, s] \),

\[
|g(x, t)| \leq h(x) + \frac{1}{x} f(x) \phi(t),
\]

whenever \( \frac{\phi(t)}{x} \leq \beta \).

Let \( k \in W_{\phi,X}[a, \infty) \) be an arbitrarily, then there exists an \( M > a \) such that for every \( x \geq M \)

\[
\rho_x(f) = \frac{1}{x} \int_a^x \phi(k(\zeta))d\mu \leq \beta.
\]
For any $x < M$, we define

$$m(x) = \sup\{|g(x,t)| : t \in X\}.$$ 

Since $g(x,.)$ is bounded on $X$, then $m(x) < \infty$. Further, if $\gamma > 0$ then the condition $\rho(\gamma k) \leq \beta$ implies

$$\int_a^s |g(\zeta, \gamma k(\zeta))| d\mu = \int_a^M |g(\zeta, \gamma k(\zeta))| d\mu + \int_M^s |g(\zeta, \gamma k(\zeta))| d\mu$$

$$\leq \int_a^M m(\zeta) d\mu + \int_M^s \{h(\zeta) + 1 + \frac{1}{\zeta} f(\zeta) \phi(\gamma k(\zeta))\} d\mu$$

$$\leq K + \|h\| + \beta \|f\|,$$

for some $K > 0$. Since, it is hold for all $s \in [a, \infty)$ then

$$\int_a^\infty s|g(\zeta, \gamma k(\zeta))| d\mu \leq K + \|h\| + \beta \|f\|,$$

untuk suatu $K > 0$.

Further, if we take $\delta = K + \|h\| + \beta \|f\|$, then

$$\|P_g(\gamma k)\| \leq \delta.$$ 

By Theorem 4.4, $P_g$ is modular bounded. □

**Corollary 4.7** Let $g(.,:) : [a, \infty) \times X \to \mathbb{R}$ be a function as in the Theorem 4.6. If $g(x,.)$ is bounded on $X$ then $P_g : W_{\phi,X}[a, \infty) \to L_1[a, \infty)$ is bounded.

**References**


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