

Scattering Theory for Higher Order Differential Operators with Sparse Random Potentials

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Abstract

In this paper, the scattering theory is studied for a pair of self-adjoint operators H_0 , elliptic operators of high order, in the space $L_2(\mathfrak{R}^d)$ and $H = H_0 + V_\omega(x)$ with $V_\omega(x)$, decaying random potential on \mathfrak{R}^d , $d \geq 3$. We prove the existence and completeness of the wave operators $W_\pm(H, H_0)$ and the coincidence of the essential spectra of the operators H_0 and H .

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1 Introduction

In the present paper we study the scattering theory for a model in perturbation theory, in particular, the existence and completeness of the wave operators $W_\pm(H, H_0)$, where the Hamiltonian H is a sum of the elliptic operator of high order

$$H_0 = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha (b_{\alpha\beta}(x) D^\beta)$$

in the space $L_2(\mathfrak{R}^d)$ as a self-adjoint operator and of $V_\omega(x)$, a random potential on \mathfrak{R}^d such that the function

$$U(x) = E(V_\omega(x))$$

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is finite for almost all $x \in \mathfrak{R}$.

As an example we consider $V_\omega(x) = q_n(\omega)\xi_n(\omega)$ where q_n are independent, identically distributed random variables with $E(q_n) < c$ and ξ_n are identically distributed, ξ_n takes only two values $\{0, 1\}$, $p_n = P(\xi_n = 1 \text{ for infinitely many } n) = 1$, $\sum p_n = \infty, p_n \rightarrow 0$ as $|n| \rightarrow \infty$; $p_n = n^{-\alpha}, \alpha > 2$ and we consider that $d \geq 3$ as the physicists expect that for low disorder and away from the band edges, random Schrodinger operators should have absolutely continuous spectrum in dimension three and higher [5].

The task of perturbation theory is to deduce information about the spectral properties of $H = H_0 + V$ from those of H_0 . It is well known that the starting point of scattering theory is that the absolutely continuous part of self-adjoint operator is stable under fairly general perturbations. There are several methods to prove the existence and completeness of the wave operators such as trace class method, smooth method, Cook's method and others [6,11,12].

It was Kirsch and Krishna [8,9] who proved that for a random potentials V_ω on $l^2(Z^d)$ and $L_2(\mathfrak{R}^d)$ it is sufficient that a certain average of V_ω goes to zero at infinity rather than V_ω itself to prove the existence of wave operators $W_\pm(H, H_0)$ where $H_0 = (-\Delta)$ using Cook's method.

In this paper we use the trace class method to prove the existence and completeness of $W_\pm(H, H_0)$ for a huge class of random Schrodinger operators with these potentials and $H_0 = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha (b_{\alpha\beta}(x) D^\beta)$ in the space $L_2(\mathfrak{R}^d)$, we deduce that some average decay is already sufficient for existence and completeness of these wave operators.

The fundamental theorem for the trace class method is the Kato-Rosenblum theorem [11,12] which state that if the difference $V = H - H_0$ belongs to the trace class, then the wave operators $W_\pm(H, H_0)$ exist and are complete.

If V is in some sense compact relative to H_0 , then the essential spectra of the operators H_0 and $H = H_0 + V$ coincide [4].

2 Preliminary Notes

We recall the definition of the wave operators, let H and H_0 on Hilbert space X be selfadjoint operators, let P^{ac} and P_0^{ac} be the projections onto the absolutely

continuous subspaces X^{ac} and X_0^{ac} , we say that the wave operators (WO) $W_{\pm}(H, H_0)$ exist if the strong limits

$$W_{\pm}(H, H_0) = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0} P_0^{ac}. \tag{2.1}$$

exist.

Each of the two operators (2.1) maps isometrically $P_0^{ac}X$ into $P^{ac}X$ and establishes a unitary equivalence between the part X_0^{ac} from $P_0^{ac}X$ (i.e. the absolutely continuous part of the operator H_0) and some part of the absolutely continuous part of the operator H .

The WO (2.1) is said to be complete if its range coincides with $P^{ac}X$. In this case the absolutely continuous parts of the operators H_0 and H are unitary equivalent. The completeness of each of the operators(2.1) is equivalent to the existence of the "inverse" the WO $W_{\pm}(H_0, H)$.

By B, B_1, B_2, B_{∞} we denote, respectively, the sets of all bounded, trace class, Hilbert-Schmidt and compact operators on X .

Lemma 2.1 [12]

If $H \in B_r, r > 0$ and $A \in B$, then $AH \in B_r$ and $HA \in B_r$.

Lemma 2.2 [12]

If $H_j \in B_{r_j}, j = 1, 2$ then $H_1H_2 \in B_r$, where $r^{-1} = r_1^{-1} + r_2^{-1}$.

In particular, the product of two Hilbert-Schmidt operators is trace class operator.

The original result of trace class scattering theory is the Kato-Rosnblum theorem [11]

Theorem 2.1

Let H_0 and H be selfadjoint operators in spaces X_0 and X , respectively, $J : X_0 \rightarrow X$ is a bounded operator and $V = HJ - JH_0 \in B_1$. Then the wave operators $W_{\pm}(H, H_0, J)$ exist and are complete.

Kuroda and Birman extend this theorem by the idea of using the resolvent [3] where in some applications we cannot use the Kato-Rosnblum theorem as V isn't trace class.

Kuroda-Birman theorem 2.2

Let H_0 and H be selfadjoint operators so that

$$(H + I)^{-1} - (H_0 + I)^{-1} \in B_1 \tag{2.2}$$

then WO $W_{\pm}(H_0, H)$ exist and are complete.

We consider the identity

$$(H + I)^{-1} - (H_0 + I)^{-1} = -(H + I)^{-1}V(H_0 + I)^{-1} \quad (2.3)$$

connecting the resolvents of the operators H_0 and H .

Now we can check that the wave operators

$$W_{\pm}(H, H_0) = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}$$

exist and are complete, that is their ranges coincide with the absolutely continuous subspace of the operator H . According to Kuroda-Birman theorem, to that end it suffices to verify that operator

$$(H + I)^{-1} - (H_0 + I)^{-1} \in B_1$$

let us now use the resolvent identity

$$(H + I)^{-1} - (H_0 + I)^{-1} = -((H + I)^{-1}(H_0 + I))((H_0 + I)^{-1}V(H_0 + I)^{-1}$$

the first factor in the right-hand side is bounded because V is bounded and

$$(H + I)^{-1}(H_0 + I) = Id - (H + I)^{-1}V.$$

So, we have to prove that the second is in trace class i.e;

$$(H_0 + I)^{-1}V(H_0 + I)^{-1} \in B_1 \quad (2.4)$$

But we have

$$(H_0 + I)^{-1}V(H_0 + I)^{-1} = ((H_0 + I)^{-1}|V|^{\frac{1}{2}})sgnV(|V|^{\frac{1}{2}}(H_0 + I)^{-1})$$

so it is necessary to show that

$$(|V|^{\frac{1}{2}}(H_0 + I)^{-1}) \in B_2. \quad (2.5)$$

With the aid of the following theorem we can test the membership in the class B_r of the operator of the Fourier type

Theorem 2.3 [12]

Let

$$|b_1(x)| \leq c(1 + |x|)^{-\alpha}, |b_2(y)| \leq c(1 + |y|)^{-\alpha}, \alpha > 0$$

then the operator T ,

$$Tf(x) = b_1(x) \int_{E^d} e^{\pm ixy} b_2(y) f(y) dy, \quad (2.6)$$

belongs to the class B_r if $r > d\alpha^{-1}$, while for $T \in B_\infty$ it is sufficient that the functions b_1, b_2 be bounded and should tend to zero at infinity.

By virtue of the generalized Weyl theorem, it is sufficient to show that

$$R(z) - R(z_0) \in B_\infty \tag{2.7}$$

to prove that the essential spectra of the operators H_0 and $H = H_0 + V$ coincide.

3 The operator of higher order

Throughout this section denote by D_j the operator $(1/i)(\partial/\partial x_j)$, and we use the standard multi-index notation: if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, is an n -tuple of nonnegative integers,

$$x^\alpha = (x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_d^{\alpha_d}) \text{ and}$$

$$D^\alpha = (-1)^{|\alpha|} \prod_{j=1}^m \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}},$$

where $|\alpha| = \sum_{j=1}^d \alpha_j$. Let t_0 is sesquilinear form with domain $D(t_0) = W^{2,m}(\mathfrak{R}^d)$ and

$$t_0(u, v) = \sum_{|\alpha|, |\beta| \leq m} \langle b_{\alpha\beta}(x) D^\beta u, D^\alpha v \rangle_{L^2},$$

where m is non-negative integer, $m \leq d$, β and α are multi-indices and $b_{\alpha\beta}(x)$ satisfies the following :

- (I) $b_{\alpha\beta}$ are bounded real measurable functions on \mathfrak{R}^d and all $b_{\alpha\beta}$ with $|\alpha| = |\beta|$ are uniformly continuous on \mathfrak{R}^d ,
- (II) $b_{\alpha\beta} = b_{\beta\alpha}$,
- (III) $\exists C_0 > 0$ such that $\sum_{|\alpha|=|\beta|=m} b_{\alpha\beta} \xi^{\alpha+\beta} \geq C_0 |\xi|^{2m}$, $\xi \in \mathfrak{R}^d$. (uniform strong ellipticity).
- (IV) $\exists C_1 > 0$ such that $\sum_{|\alpha|=|\beta|=m} \{b_{\alpha\beta} + v_0 + v(x)\} \xi^{\alpha+\beta} \geq C_1 |\xi|^{2m}$, $\xi, x \in \mathfrak{R}^d$.

The principle part of t_0 will be denoted by:

$$s_0(u, v) = \sum_{|\alpha|=|\beta|=m} \langle b_{\alpha\beta}(x) D^\beta u, D^\alpha v \rangle,$$

$u, v \in W^{2,m}(\mathfrak{R}^d)$. Note that each of t_0 and s_0 are bounded forms. We shall prove that t_0 is positive form. For this we need the following:

Lemma 3.1 Assume that s_0 has constant coefficient, then for all $u \in W^{2,m}(\mathfrak{R}^d)$

$$s_0(u) \geq \nu C_0 \|u\|_m^2,$$

where ν depend on d, m .

Theorem 3.2 If s_0 has constant coefficients. Then for all $u \in W^{2,m}(\mathfrak{R}^d)$ and some constant $\varrho > 0$

$$t_0(u) \geq \varrho \|u\|^2.$$

Theorem 3.3 Assume that $t_0 = s_0$ that the coefficients $b_{\alpha\beta}(x)$ are uniformly continuous, $|\alpha| = |\beta| = m$. Then, for all $u \in W^{2,m}(\mathfrak{R}^d)$,

$$t_0(u) \geq \nu C_0 \|u\|^2,$$

One can generalize the above theorem if the condition $t_0 = s_0$ is omit by the following:

Corollary 3.4 The above Theorem remains true without the assumption $t_0(u) = s_0$.

Take $b_{\alpha\beta}$ are constant then $t_0(u, v)$ is non-negative symmetric closed form with $W^{2,m}(\mathfrak{R}^d)$. Let H_0 be the non-negative self-adjoint operator associated with $t_0(u, v)$ in the sense of Friedrichs (see [7]). Namely, H_0 is the unique non-negative self-adjoint operator such that:

$$t_0(u, v) = \langle H_0 u, v \rangle, \quad u \in D(H_0), v \in D(t_0),$$

$$H_0 = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha (b_{\alpha\beta} D^\beta). \tag{3.1}$$

Let P be the polynomial given by:

$$P(x) = \sum_{|\alpha|, |\beta| \leq m} b_{\alpha\beta} x^{\alpha+\beta},$$

$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, + \dots, \alpha_d + \beta_d)$. It is given well-known that:

$$D(H_0) = W^{2,2m}(\mathfrak{R}^d) \quad H_0 u = P(D)u, \quad u \in W^{2,2m}(\mathfrak{R}^d).$$

The spectrum $\sigma(H_0)$ of H_0 is equal to $[\lambda_{min}, \infty[$ where $\lambda_{min} = \inf_{x \in \mathfrak{R}^d} P(x)$. We use the following definition (see [2,10]).

Definition 3.5 $\lambda \in \mathfrak{R}$ is said to be critical value of the polynomial P if there exists $x \in \mathfrak{R}^d$ such that $P(x) = \lambda$ and $gradP(x) = 0$.

The set e_0 of all critical values of P is closed set of measure zero and H_0 is absolutely continuous in $\mathfrak{R} - e_0$.

One have H is the realization of the differential operator $H_0 + V_\omega$,

$$H = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha (b_{\alpha\beta} D^\beta) + V_\omega, \tag{3.2}$$

which is non-negative self-adjoint operator.

Let $F : L_2(\mathfrak{R}^d) \longrightarrow L_2(\Xi^d)$ be the Fourier transformation and Ξ^d the dual space of \mathfrak{R}^d , i.e., F is the integral operator with kernel $(2\pi)^{-d/2} \exp(\langle -ix, \xi \rangle)$. From the properties of F and the ellipticity of the non-negative self-adjoint H_0 one have, for $\mu \in \rho(H_0), \mu < 0$, the resolvent operator $(H_0 - \mu)^{-1}$ is bounded and

$$F(H_0 - \mu)^{-1} = (P(x) - \mu)^{-1}F = \left(\sum_{|\alpha|, |\beta| \leq m} b_{\alpha\beta} x^{\alpha+\beta} - \mu \right)^{-1} F$$

4 Main Results

We set $H = H_0 + V_\omega; H_0 = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha (b_{\alpha\beta}(x) D^\beta)$ on the space $L_2(\mathfrak{R}^d)$. Suppose $V_\omega(x)$ is a random potential on \mathfrak{R}^d such that the function

$$U(x) = E(V_\omega(x))$$

is finite for almost all $x \in \mathfrak{R}$. We want to prove the existence and completeness of the wave operators $W_\pm(H, H_0)$. By Birman theorem we need to prove (2.4). Now we use the average condition.

It sufficient to prove that

$$(H_0 + I)^{-1} V_\omega (H_0 + I)^{-1} \in B_1$$

for almost all ω , which certainly true if the expectation of the quantity is in trace class

$$\begin{aligned} & E(H_0 + I)^{-1} V_\omega (H_0 + I)^{-1} \\ &= (H_0 + I)^{-1} E(V_\omega) (H_0 + I)^{-1} \\ &= (H_0 + I)^{-1} U (H_0 + I)^{-1} \end{aligned}$$

So, we need to prove that

$$(H_0 + I)^{-1} U (H_0 + I)^{-1} \in B_1 \tag{4.1}$$

or by (2.5) (similarly) we need to prove that

$$(|U|^{\frac{1}{2}} (H_0 + I)^{-1}) \in B_2. \tag{4.2}$$

Theorem 4.1

If $V_\omega(x)$ is a random potential on \mathfrak{R}^d such that the function

$$U(x) = E(V_\omega(x))$$

is finite for almost all $x \in \mathfrak{R}$ and satisfy (4.1) then the wave operators W_{\pm} for V_{ω} exist and are complete.

There are many examples which illustrate this theorem.

As an example we consider $V_{\omega}(x) = q_n(\omega)\xi_n(\omega)$ where q_n are independent, identically distributed random variables with $E(q_n) < c$ and ξ_n are identically distributed, ξ_n takes only two values $\{0, 1\}$, $p_n = P(\xi_n = 1$ for infinitely many $n) = 1$, $\sum p_n = \infty, p_n \rightarrow 0$ as $|n| \rightarrow \infty; p_n = n^{-\alpha}, \alpha > 2$ for large n .

To prove the existence and completeness of the wave operators $W_{\pm}(H, H_0)$ in this case where $H = H_0 + V_{\omega}; H_0 = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\alpha|} D^{\alpha} (b_{\alpha\beta}(x) D^{\beta})$ we use (4.2), by virtue of (2.3) the inclusion (4.2) is equivalent to the fact that $(|U|^{\frac{1}{2}} F^{-1} \left(\sum_{|\alpha|, |\beta| \leq m} b_{\alpha\beta} x^{\alpha+\beta} + I \right)^{-1}) \in B_2$. The last operator is integral and, moreover, its kernel is

$$(2\pi)^{-\frac{d}{2}} |U|^{\frac{1}{2}} e^{ixy} \left(\sum_{|\alpha|, |\beta| \leq m} b_{\alpha\beta} x^{\alpha+\beta} + I \right)^{-1}.$$

using theorem (2.3) and (III) $b_1 = |U|^{\frac{1}{2}} = (E(q_n(\omega))^{\frac{1}{2}} (p_n)^{\frac{1}{2}} \leq c(1 + |n|)^{-\frac{\alpha}{2}}$ and $b_2 = \left(\sum_{|\alpha|, |\beta| \leq m} b_{\alpha\beta} x^{\alpha+\beta} + I \right)^{-1} \leq \left(\sum_{|\alpha|, |\beta| = m} b_{\alpha\beta} x^{\alpha+\beta} + I \right)^{-1} \leq c(x^{2m} + I)^{-1} \leq c(x + I)^{-2m}$

and $2 > \frac{d}{2m}$ which tends to $4m > d$ hence $(|U|^{\frac{1}{2}}(H_0 + I)^{-1})$ and also $(H_0 + I)^{-1}|U|^{\frac{1}{2}}$ are Hilbert-Schmidt operator, then by lemma (2.2) we deduce that

$$(H_0 + I)^{-1}U(H_0 + I)^{-1} \in B_1$$

which complete the proof of the existence and completeness of $W_{\pm}(H, H_0)$.

To prove that the essential spectra of the operator H_0 and $H = H_0 + V$ coincide it suffices to show that V is compact relative to H_0 . so by using the generalized Weyl theorem and the resolvent identity, it is sufficient to show that

$$(H_0 + I)^{-1}U(H_0 + I)^{-1} \in B_{\infty}. \tag{4.3}$$

Since $(H_0 + I)^{-1}$ is a bounded operator, then we want to show that

$$U(H_0 + I)^{-1} \in B_{\infty}$$

this inclusion is equivalent to the fact that $U \left(\sum_{|\alpha|, |\beta| \leq m} b_{\alpha\beta} x^{\alpha+\beta} + I \right)^{-1} \in B_{\infty}$. the last operator is integral and its kernel is

$$(2\pi)^{-\frac{d}{2}} U e^{ixy} \left(\sum_{|\alpha|, |\beta| \leq m} b_{\alpha\beta} x^{\alpha+\beta} + I \right)^{-1}.$$

therefore, from the conditions on U the inclusion (4.3) follows from theorem(2.3) since $b_1 = E(q_n)n^{-\alpha}$, $b_2 = \left(\sum_{|\alpha|,|\beta|\leq m} b_{\alpha\beta}x^{\alpha+\beta} + I \right)^{-1}$ tends to zero at infinity.

Let us formulate the result obtained.

Lemma 4.2

Let $H_0 = \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\alpha|} D^\alpha (b_{\alpha\beta}(x) D^\beta)$, $H = H_0 + V$ where $V_\omega(x) = q_n(\omega)\xi_n(\omega)$ where q_n are independent, identically distributed random variables with $E(q_n) < c$ and ξ_n are identically distributed, ξ_n takes only two values $\{0, 1\}$, $p_n = P(\xi_n = 1 \text{ for infinitely many } n) = 1$, $\sum p_n = \infty$, $p_n \rightarrow 0$ as $|n| \rightarrow \infty$; $p_n = n^{-\alpha}$, $\alpha > 2$ for large n , $4m > d$, $m \geq 1$, then the wave operators exist and are complete. In particular, the absolutely continuous part of H is unitarily equivalent to the operator H_0 . Moreover, the essential spectra of the operators H_0 and H are coincide.

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