

On $[w]_{\sigma,\theta}^L$ -Lacunary Asymptotically Equivalent Sequences

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Abstract. This paper presents new definitions which are a natural combination of the definition for asymptotically equivalence and $[w]_{\sigma,\theta}$ -statistically convergence. Using this definitions we have proved the st- $[w]_{\sigma,\theta}^L$ -asymptotically equivalence analogues of Fridy and Orhan's theorems in [8] and analogues results of Das and Patel in [2].

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1. INTRODUCTION

Let l_∞ and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ with the usual norm $\|x\| = \sup_k |x_k|$. A sequence $x = (x_k) \in l_\infty$ is said to be almost convergent if all of its Banach limits coincide. Let \widehat{c} denote the space of all almost convergent sequences. Lorentz [4] proved that

$$\widehat{c} = \left\{ x \in l_\infty : \lim_m d_{mn}(x) \text{ exists uniformly in } n \right\}$$

where

$$d_{mn}(x) = \frac{x_n + x_{n+1} + \dots + x_{n+m}}{m+1}.$$

The space $[\widehat{c}]$ is of strongly almost convergent sequences was introduced by Maddox [6] as follows:

$$[\widehat{c}] = \left\{ x \in l_\infty : \lim_m d_{mn}(|x - le|) \text{ exists uniformly in } n \text{ for some } l \in \mathbb{C} \right\}$$

where $e = (1, 1, \dots)$.

Let σ be one-to-one mapping of the set of positive integers into itself such that $\sigma^k(n) = \sigma(\sigma^{k-1}(n))$, $k = 1, 2, 3, \dots$. A continuous linear functional φ on l_∞ is said to be an invariant mean or a σ -mean if and only if

- (1) $\varphi \geq 0$ when the sequence has $x_n \geq 0$ for all n
- (2) $\varphi(e) = 1$ where $e = (1, 1, \dots)$ and
- (3) $\varphi(x_{\sigma(n)}) = \varphi(x)$ or all $x \in l_\infty$.

For a certain kinds of mapping σ every invariant mean φ extends the limit functional on space c , in the sense that $\varphi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$ where V_σ is the bounded sequences all of whose σ -means are equal.

It can be shown [12] that

$$V_\sigma = \left\{ x \in l_\infty : \lim_k t_{km}(x) = Le \text{ uniformly in } m \text{ for some } L = \sigma - \lim x \right\}$$

where

$$t_{km}(x) = \frac{x_m + x_{\sigma(m)} + \dots + x_{\sigma^k(m)}}{k + 1}.$$

We say that a bounded sequence $x = (x_k)$ is σ -convergent if and only if $x \in V_\sigma$ such that $\sigma^k(m) \neq m$ for all $m \geq 0$, $k \geq 1$.

In [11] Mursaleen, Gaur and Chishti introduced the following sequence space, which is generalized the sequence space $[w]$ of Das and Sahoo [3],

$$[w]_\sigma = \left\{ x = (x_k) : \frac{1}{n+1} \sum_{k=0}^n |t_{km}(x - L)| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ uniformly in } m \text{ for some } L \right\}$$

$$\text{where } t_{km}(x) = \frac{x_m + x_{\sigma(m)} + \dots + x_{\sigma^k(m)}}{k+1}.$$

The idea of statistical convergence was introduced by Fast [5] and studied by various authors (see [9], [7], [15]). A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S - \lim x = L$ $x_k \rightarrow L(S)$ and S denotes the set of all statistically convergent sequences.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, \dots$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r .

In 1993 Marouf [10] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003 Patterson [13] extend these concepts by presenting an asymptotically statistically equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. In addition to the extensions above Savaş and Patterson incorporated lacunary sequences into these notions in [1]. Furthermore, asymptotically equivalence was studied in [14].

This paper presents new definitions which are a natural combination of the definition for asymptotically equivalence and $[w]_{\sigma, \theta}$ -statistically convergence. In addition to these definitions, natural inclusion theorems shall also be presented.

2. DEFINITIONS AND NOTATIONS

Definition 1. (Marouf [10]) *Two nonnegative sequences x, y are said to be asymptotically equivalent if*

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

Definition 2. (Patterson [13]) *Two nonnegative sequences x, y are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0$$

(denoted by $x \stackrel{SL}{\sim} y$) and simply asymptotically statistical equivalent, if $L=1$.

Definition 3. *A sequence $x = (x_n)$ is said to be $[w]_{\sigma, \theta}$ -statistically convergent or st - $[w]_{\sigma, \theta}$ -convergent to L if for every $\varepsilon > 0$*

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |t_{km}(x) - L| \geq \varepsilon\}| = 0$$

uniformly in $m = 1, 2, 3, \dots$. In this case, we write $st - [w]_{\sigma, \theta} - \lim x = L$ or $x_k \rightarrow L(st - [w]_{\sigma, \theta})$. We denote the set of these sequences by $st - [w]_{\sigma, \theta}$.

Definition 4. A sequence $x = (x_n)$ is said to be $[w]_{\sigma, \theta}$ -convergent to L if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |t_{km}(x) - L| = 0$$

uniformly in $m = 1, 2, 3, \dots$. In this case, we write $[w]_{\sigma, \theta} - \lim x = L$ or $x_k \rightarrow L([w]_{\sigma, \theta})$. We denote the set of these sequences by $[w]_{\sigma, \theta}$.

In addition, these definitions were also extended to incorporate lacunary sequence concept. Following these results Savaş and Patterson [1] introduce new definitions, namely σ -asymptotically lacunary statistical equivalent of multiple L and strong σ -asymptotically lacunary equivalent of multiple L .

Definition 5. (Savaş and Patterson [1]) Let θ be a lacunary sequence; the two nonnegative sequences x, y are $S_{\sigma, \theta}$ -asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| \geq \varepsilon \right\} \right| = 0$$

uniformly in $m = 1, 2, 3, \dots$ (denoted by $x \stackrel{S_{\sigma, \theta}}{\sim} y$) and simply $S_{\sigma, \theta}$ -asymptotically equivalent, if $L=1$.

Definition 6. (Savaş and Patterson [1]) Two nonnegative sequences x, y are strong σ -asymptotically lacunary equivalent of multiple L provided that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L \right| = 0$$

uniformly in $m = 1, 2, 3, \dots$ (denoted by $x \stackrel{N_{\sigma, \theta}}{\sim} y$) and simply strong σ -asymptotically lacunary equivalent, if $L=1$.

Following this definitions which are given above, we shall now introduce following new notions $[w]_{\sigma}$ -asymptotically equivalence, $st - [w]_{\sigma}^L$ -asymptotically equivalent of multiple L , asymptotically equivalent of multiple L and $[w]_{\sigma, \theta}^L$ -asymptotically equivalent of multiple L .

Definition 7. Two nonnegative sequences x, y are said to be $[w]_{\sigma}$ -asymptotically equivalent if

$$\lim_k \frac{t_{km}(x)}{t_{km}(y)} = 1$$

uniformly in $m = 1, 2, 3, \dots$, where $t_{km}(x) = \frac{x_m + x_{\sigma(m)} + \dots + x_{\sigma^k(m)}}{k+1}$ (denoted by $x \stackrel{[w]_{\sigma}}{\sim} y$).

Definition 8. Two nonnegative sequences x, y are $st - [w]_\sigma^L$ -asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| = 0$$

uniformly in $m = 1, 2, 3, \dots$ (denoted by $x \overset{st-[w]_\sigma^L}{\sim} y$) and simply $[w]_\sigma$ -asymptotically statistical equivalent, if $L=1$.

Definition 9. Two nonnegative sequences x, y are $st - [w]_{\sigma,\theta}^L$ -asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| = 0$$

uniformly in $m = 1, 2, 3, \dots$ (denoted by $x \overset{st-[w]_{\sigma,\theta}^L}{\sim} y$) and simply $st - [w]_{\sigma,\theta}$ -asymptotically equivalent, if $L=1$.

Definition 10. Two nonnegative sequences x, y are $[w]_{\sigma,\theta}^L$ -asymptotically equivalent of multiple L provided that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| = 0$$

uniformly in $m = 1, 2, 3, \dots$ (denoted by $x \overset{[w]_{\sigma,\theta}^L}{\sim} y$) and simply $[w]_{\sigma,\theta}$ -asymptotically equivalent, if $L=1$.

If we take $\sigma(n) = n + 1$ then $[w]_\sigma$ -asymptotically equivalence, $st - [w]_\sigma^L$ -asymptotically equivalence, $st - [w]_{\sigma,\theta}^L$ -asymptotically equivalence and $[w]_{\sigma,\theta}^L$ -asymptotically equivalence reduce $[w]$ -asymptotically equivalence, $\widehat{st} - [w]^L$ -asymptotically equivalence, $\widehat{st} - [w]_\theta^L$ -asymptotically equivalence and $[w]_\theta^L$ -asymptotically equivalence; respectively.

3. MAIN RESULTS

Theorem 1. Let $\theta = \{k_r\}$ be a lacunary sequence then

- (i) If $x \overset{[w]_{\sigma,\theta}^L}{\sim} y$ then $x \overset{st-[w]_{\sigma,\theta}^L}{\sim} y$,
- (ii) If $x, y \in l_\infty$ and $x \overset{st-[w]_{\sigma,\theta}^L}{\sim} y$ then $x \overset{[w]_{\sigma,\theta}^L}{\sim} y$,
- (iii) $x \overset{st-[w]_{\sigma,\theta}^L}{\sim} y \cap l_\infty = x \overset{[w]_{\sigma,\theta}^L}{\sim} y \cap l_\infty$.

Proof. (i) If $\varepsilon > 0$ and $x \overset{[w]_{\sigma,\theta}^L}{\sim} y$ then

$$\sum_{k \in I_r} \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \sum_{\substack{k \in I_r \\ \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon}} \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \left| \left\{ k \in I_r : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right|.$$

Therefore $x \underset{\sim}{\overset{st-[w]_{\sigma,\theta}^L}{\sim}} y$.

(ii) Suppose x, y are in l_∞ and $x \underset{\sim}{\overset{st-[w]_{\sigma,\theta}^L}{\sim}} y$. Then we can assume that

$$\left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \leq M \quad \text{for all } k \text{ and } m.$$

Given $\varepsilon > 0$

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| &= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon}} \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| < \varepsilon}} \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \\ &\leq \frac{M}{h_r} \left| \left\{ k \in I_r : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| + \varepsilon. \end{aligned}$$

Therefore $x \underset{\sim}{\overset{[w]_{\sigma,\theta}^L}{\sim}} y$.

(iii) This immediately follows from (i) and (ii). ■

In order to show that the converse of Theorem 1 (i) is not generally true, we now give the following example.

Example 1. Let $\theta = \{k_r\}$ be given and define $x = (x_i)$ to be

$$1, k_{r-1} + 2^2, 2k_{r-1} + 3^2, 3k_{r-1} + 4^2, \dots, \left(\left[\sqrt{h_r} \right] - 1 \right) k_{r-1} + \left[\sqrt{h_r} \right]^2$$

for $i = \sigma^k(m), k = k_{r-1} + 1, k_{r-1} + 2, \dots, k_{r-1} + \left[\sqrt{h_r} \right]; m \geq 1$ and $(x_i) = 1$ otherwise (where $[\]$ denotes the greatest integer function) and $y_k = 1$ for all k . Note that x is not bounded and $t_{km}(x)$ be

$$1, 2, \dots, \left[\sqrt{h_r} \right];$$

for $k = k_{r-1} + 1, k_{r-1} + 2, \dots, k_{r-1} + \left[\sqrt{h_r} \right]; m \geq 1$ and $t_{km}(x) = 1$ otherwise and $t_{km}(y) = 1$ and for all k and m .

Further, for $\varepsilon > 0$, we have

$$\frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| = \frac{[\sqrt{h_r}]}{h_r} \rightarrow 0 \text{ as } r \rightarrow \infty,$$

i.e., $x \overset{st-[w]_{\sigma,\theta}^L}{\sim} y$. On the other hand,

$$\frac{1}{h_r} \sum_{k \in I_r} \left| \frac{t_{km}(x)}{t_{km}(y)} - 1 \right| = \frac{1}{h_r} \frac{[\sqrt{h_r}]([\sqrt{h_r}] - 1)}{2} \rightarrow \frac{1}{2} \neq 0 \text{ as } r \rightarrow \infty,$$

hence $x \not\overset{[w]_{\sigma,\theta}^L}{\sim} y$ (x, y are not simply $[w]_{\sigma,\theta}$ -asymptotically equivalent).

Lemma 1. Suppose that for given $\varepsilon_1 > 0$ and every $\varepsilon > 0$ there exists n_0 and m_0 such that

$$\frac{1}{n} \left| \left\{ 0 \leq k \leq n - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| < \varepsilon_1$$

for all $n \geq n_0$ and $m \geq m_0$ then $x \overset{st-[w]_{\sigma}^L}{\sim} y$.

Proof. Let ε_1 be given. For every $\varepsilon > 0$, choose n'_0 and m_0 such that

$$(3.1) \quad \frac{1}{n} \left| \left\{ 0 \leq k \leq n - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| < \frac{\varepsilon_1}{2}$$

for all n''_0 and $m \geq m_0$. It is sufficient to prove that there exists n''_0 such that for $n \geq n''_0$ and $0 \leq m \leq m_0$

$$(3.2) \quad \frac{1}{n} \left| \left\{ 0 \leq k \leq n - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| < \varepsilon_1.$$

If we let $n_0 = \max \{n'_0, n''_0\}$ (3.2) will be true for $n > n_0$ and for all m . Once m_0 has been chosen, m_0 is fixed, so

$$\left| \left\{ 0 \leq k \leq m_0 - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| = M.$$

Now taking $0 \leq m \leq m_0$ and $n > m_0$, by (3.1) we have

$$\begin{aligned} & \frac{1}{n} \left| \left\{ 0 \leq k \leq n - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| \\ &= \frac{1}{n} \left| \left\{ 0 \leq k \leq m_0 - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| + \\ &+ \frac{1}{n} \left| \left\{ m_0 \leq k \leq n - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| \end{aligned}$$

$$= \frac{M}{n} + \frac{1}{n} \left| \left\{ m_0 \leq k \leq n - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| < \frac{M}{n} + \frac{\varepsilon}{2}.$$

Thus for n sufficiently large

$$\frac{1}{n} \left| \left\{ 0 \leq k \leq n - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| < \frac{M}{n} + \frac{\varepsilon}{2} < \varepsilon_1.$$

Thus (3.2) holds. This yields the result. ■

Theorem 2. $x \underset{\sim}{\sim}^{st-[w]_{\sigma,\theta}^L} y \iff x \underset{\sim}{\sim}^{st-[w]_{\sigma}^L} y$ for every θ .

Proof. Let $x \underset{\sim}{\sim}^{st-[w]_{\sigma,\theta}^L} y$, then from Definition 9 assures us that, given $\varepsilon_1 > 0$ there exists $\varepsilon > 0$ and L such that

$$(3.3) \quad \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| < \varepsilon_1$$

for $r \geq r_0$ and $m = k_{r-1} + 1 + u$ where $u \geq 0$. Let $n \geq h_r$ and write $n = ih_r + t$ with $0 \leq t \leq h_r$ and i an integer.

Since $n \geq h_r$ and $i \geq 1$, we obtain the following by (3.3):

$$\begin{aligned} & \frac{1}{n} \left| \left\{ 0 \leq k \leq n - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| \\ & \leq \frac{1}{n} \left| \left\{ 0 \leq k \leq (i + 1)h_r - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| \\ & = \frac{1}{n} \sum_{j=0}^i \left| \left\{ jh_r \leq k \leq (j + 1)h_r - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| \\ & < \frac{(i + 1)h_r}{n} \varepsilon_1 < \frac{2ih_r\varepsilon_1}{n} \text{ for } i \geq 1. \end{aligned}$$

For $\frac{h_r}{n} \leq 1$, since $\frac{ih_r}{n} \leq 1$. Therefore

$$\frac{1}{n} \left| \left\{ 0 \leq k \leq n - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| < 2\varepsilon_1.$$

Hence Lemma 1 implies $x \underset{\sim}{\sim}^{st-[w]_{\sigma,\theta}^L} y \Rightarrow x \underset{\sim}{\sim}^{st-[w]_{\sigma}^L} y$.

We now show that $x \underset{\sim}{\sim}^{st-[w]_{\sigma}^L} y \Rightarrow x \underset{\sim}{\sim}^{st-[w]_{\sigma,\theta}^L} y$. Let $x \underset{\sim}{\sim}^{st-[w]_{\sigma}^L} y$, then from Definition 8 given $\varepsilon_1 > 0$ there exists $\varepsilon > 0$ and L such that

$$(3.4) \quad \frac{1}{n} \left| \left\{ 0 \leq k \leq n - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| < \varepsilon_1.$$

Let $n \geq h_r$ and write $n = ih_r + t$ with $0 \leq t \leq h_r$ and i an integer. Since $n \geq h_r$ and $n \leq (i + 1)h_r$, we obtain

$$\begin{aligned}
 & \frac{1}{n} \left| \left\{ 0 \leq k \leq n-1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| \\
 & \geq \frac{1}{n} \left| \left\{ 0 \leq k \leq ih_r - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| \\
 & \geq \frac{1}{(i+1)h_r} \left| \left\{ 0 \leq k \leq ih_r - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| \\
 & \geq \frac{1}{2ih_r} \left| \left\{ 0 \leq k \leq ih_r - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| \\
 & = \frac{1}{2ih_r} \sum_{j=1}^i \left| \left\{ (j-1)h_r \leq k \leq jh_r - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| \\
 & = \frac{1}{2h_r} \left| \left\{ 0 \leq k \leq h_r - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right|.
 \end{aligned}$$

Therefore, by (3.4) we have

$$\frac{1}{2h_r} \left| \left\{ 0 \leq k \leq h_r - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| \leq \frac{1}{n} \left| \left\{ 0 \leq k \leq n-1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| < \varepsilon_1,$$

that is

$$\frac{1}{h_r} \left| \left\{ 0 \leq k \leq h_r - 1 : \left| \frac{t_{km}(x)}{t_{km}(y)} - L \right| \geq \varepsilon \right\} \right| \leq 2\varepsilon_1.$$

Hence we obtain $x \overset{st-[w]_\sigma^L}{\sim} y \Rightarrow x \overset{st-[w]_{\sigma,\theta}^L}{\sim} y$. This completes the proof.

■

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