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Periodic Solutions of the Duffing Equation

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Abstract

The Duffing equation will be studied under a variety of different values for the coupling parameters in the equation. The general expression for the Duffing equation can be written in the form of a coupled first order system. Using this form for the system, the periodic structure of the equation can be studied through the use of phase plots. Numerous graphs corresponding to this are shown. In addition to this the period of the phase plot plus the inverse of the angular frequency are presented. A linear relationship is observed for this differential equation for different values of the coupling parameter.

Keywords: Duffing equation, phase plot, period

I. Introduction

The differential equation which describes a nonlinear oscillator first introduced by Duffing **[1-3]** with a cubic stiffness constant has become a very

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common example of a nonlinear oscillator. This equation permits the description of a hard spring and remains of continuing interest. For example, in a family of planar maps depending on parameters, the onset of chaos typically occurs at the parameter values for which the stable and unstable manifolds of a stable point come into contact tangentially. This method for creation of transversal homoclinic points and related dynamics can be established for the forced Duffing equation [**4,5**]. The general form of the equation which will be studied here has the form

$$\ddot{x}(t) + \delta \dot{x}(t) - x(t) + \beta x^3(t) = f(t). \tag{1a}$$

where for our purposes, $f(t)$ is one of the following two functions

$$f(t) = \gamma \cos \omega t, \quad f(t) = \gamma \sin \omega t. \tag{1b}$$

This provides a simple model for a number of nonlinear systems. It can be represented as a pair of coupled first order equations expressed as

$$\begin{gathered} \dot{x}(t) = v(t), \\ \dot{v}(t) = -\delta v(t) + x(t) - \beta (x(t))^3 + f(t). \end{gathered} \tag{2}$$

This representation of the Duffing equation allows it to be defined as a function in terms of the pair $(x(t), v(t))$. This provides a way to produce a phase plot since such a plot is a graph of the relationship between $x(t)$ and $v(t)$. The phase plot can be used to give the period of the specific motion generated by the equation for specific values of the parameters in (1). The solution plot is often not regular and often appears chaotic. The period can provide some intriguing information as to the nature of the equation. The study carried out here begins by picking values for the constants, some of which were not changed during different trials, such as δ which was fixed to be $\delta = 0.1$. This ensures that the equation itself remains stable. The parameter γ which gives a homogeneous equation when set to zero, was taken to be 0.1 so a driving force is present. Finally the parameter β was set to be one so the system represents a hard spring. Since β is the constant in front of the nonlinear term when it is put to small values, a more or less linear approximation of the Duffing equation results. The one term that was changed throughout all the trials was the ω parameter. As ω changes, so does the effect of the damping oscillator, so the larger the term, the larger the damping effect is. Here three sample trials were produced using different values for the constant ω term. The Duffing equation's phase plots were produced and studied to predict a relationship between the two parameters T and ω.

II. Results.

The first phase plot for the Duffing equation has been evaluated at $\omega = 1/2$. This particular phase plot seems a bit chaotic at first when observed over $0 < t < 100$, but when it is examined on the interval $150 < t < 150 + (2\pi/\omega)$, as seen in Figure 1, we can see a simple loop that does not close completely. This represents a quasi period of the Duffing equation due to the damping term in ω, but since it is very close to being a closed loop an appropriate approximation of the period may still be a valid approximation to the period of the oscillator. This particular period was approximated using the graph and was evaluated to be 12.57, which is equal to about 4π. This is interesting because the period follows a simple equation which applies to the determination of periods in simple harmonic motion which is given by

$$T = 2\pi/\omega, \tag{3}$$

where T is the period. When $\omega = 1/2$, we get $T = 4\pi$ from equation (3). It should be noted that this is the same for Duffing equations with a sine driving term. Numerical results for T as a function of ω are given in Table 1 for both types of driving terms (1 b).

In Figure 2, we have a phase plot for the Duffing equation evaluated at $\omega = 1$. At first glance the phase plot is becoming more organized relative to the previous graph over $0 < t < 100$ and as we examine it from $150 < t < 150 + (2\pi/\omega)$, as in Figure 2, we see one simple loop that does not close better than shown in Figure 1. This period was approximated using the graph and was evaluated to be about 2π. Using equation (3), we would predict a value of 2π. In this case, the numerical and theoretical results match each other. If ω is allowed to approach small values, the force of the damping oscillator gets smaller which does not create great changes in the period of the phase plot.

In Figure 3, we have the phase plot of the Duffing equation evaluated with $\omega = 5$. The phase plot is much more organized than the previous phase plots evaluated at $0 < t < 100$, and as we evaluate it from $150 < t < 150 + (1.9\pi/\omega)$, as seen in Figure 3, again a simple loop that does not completely close on itself is produced. This trend is due to the combined effect of the damping oscillator and nonlinear term.

Figure 4 shows a best fit plot of the Duffing's equation's period in terms of inverse ω. This particular plot was desired because the slope can tell us about the behavior of the Duffing equation evaluated at different ω values. In Figure 5, a best fit plot of various inverse ω values and their respective periods for the Duffing equation using the sine driving term instead of cosine term is shown. The slope of both best fit plots should be about 2π if it follows (3). From the best fit plot we get an equation for a line to be $6.28317\omega^{-1}$ for the cosine driving term. which is close to $2\pi\omega^{-1}$. For the sine driving term we get $6.2834\omega^{-1}$. Both of these equations can be approximated by the result $2\pi\omega^{-1}$. This means that in general, even with the deviations of higher ω's, both follow the linear harmonic oscillator formula. When Figure 4 and 5 are juxtaposed, a slight deviation is noticed, meaning that both don't behave exactly the same, but are similar.

The Figure 5 shows a best fit plot of various inverse ω and their respected periods for the Duffing equation using a sine driving term instead of the cosine term. This particular plot was desired because the slope can tell us about the behavior of the Duffing equation evaluated at different ω values. The slope should be about 2π if it follows formula (3), evaluated using the best fit plot we get an equation of the line to be $6.2834\omega^{-1}$, which is about $2\pi\omega^{-1}$. This means that in general even with the deviations of higher ω values it generally follows the linear T formula, and does also follow very closely to the Duffing equation with the cosine driving term, but doesn't match exactly.

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Table 1: Period T_{cosine} and T_{sine} as a function of ω for Duffing equation (1) with a cosine and sine driving term.

ω	0.5	0.7	0.8	0.93	1.0	1.1	1.571	2	2.5	3	5	10	15
T_c	12.566	8.976	7.854	6.756	6.283	5.712	4.0	3.142	2.153	2.094	1.23	0.601	0.456
T_s	12.566	8.128	7.854	6.771	6.283	5.712	4.0	3.142	2.153	2.094	1.27	0.601	0.44

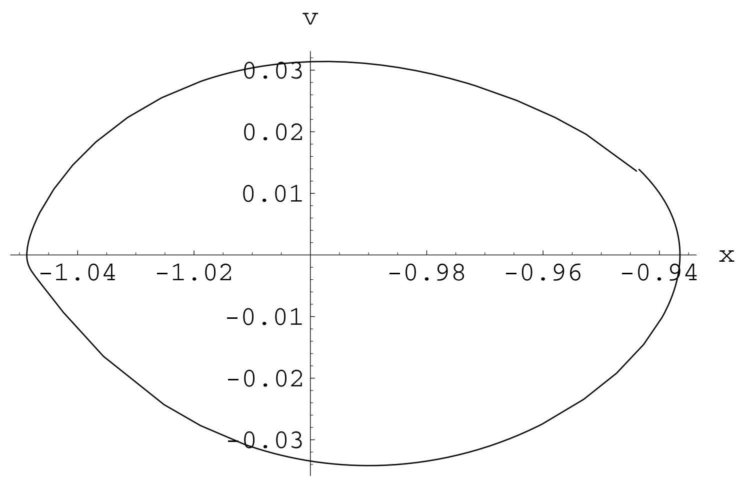

Figure 1: Plot with $\omega = 1/2$ for $150 < t < 150 + (2\pi/\omega)$ displaying a quasi-period of the Duffing equation that can be approximated as a tightly bound closed loop.

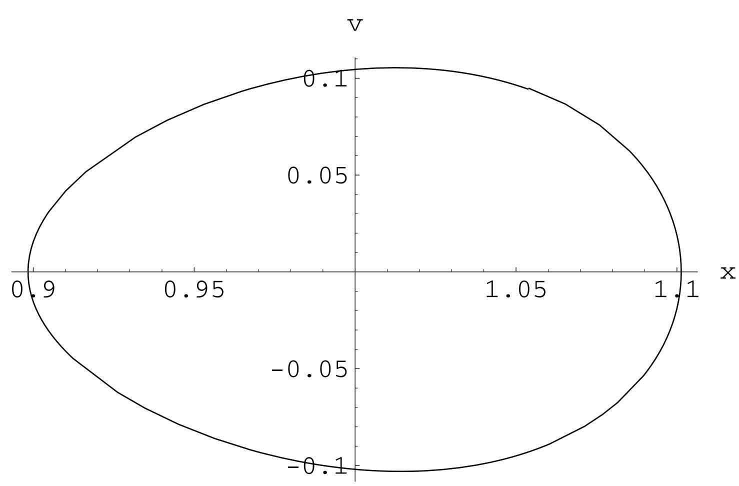

Figure 2: Plot with $\omega = 1$ and $150 < t < 150 + (2\pi/\omega)$ displaying one tight closed loop representing a full period.

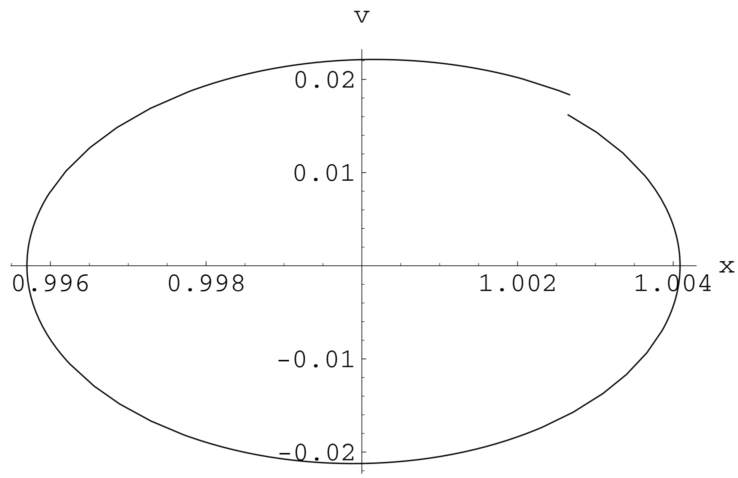

Figure 3: Plot with $\omega = 5$ for $150 < t < 150 + (2\pi/\omega)$ displaying one loop that does not close completely, a quasi period, meaning that a clear period could not be represented, but can be approximated.

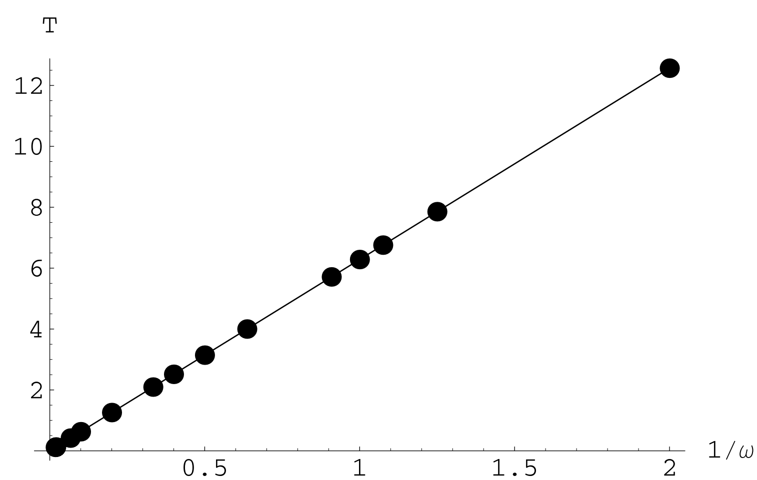

Figure 4: Plot with best fit line $T = 6.28317\omega^{-1}$ for Duffing equation with cosine driving term representing period versus ω^{-1} in order to show that when plotted, the best fit line shows a slope of 6.28317 or about 2π.

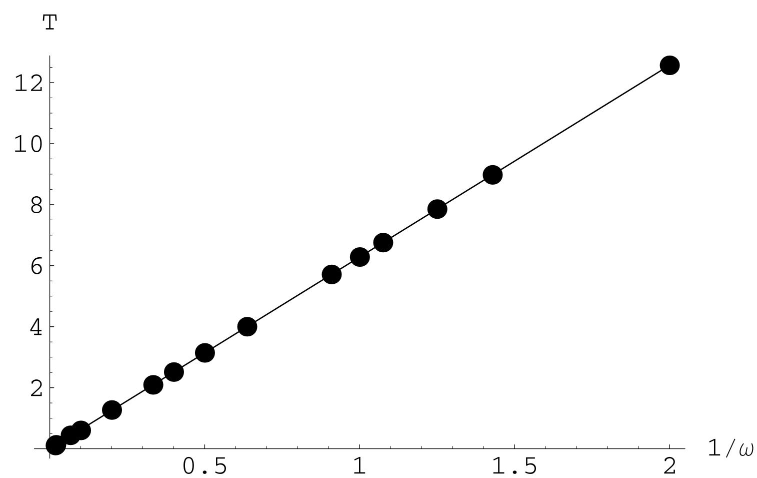

Figure 5: Plot of best fit line $T = 6.2834\omega^{-1}$ for the Duffing equation with a sine driving term representing period versus ω^{-1} in order to show that the best fit line approximates the slope at 6.2834 or about 2π.