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Continuity Properties of Potentials on Spaces of Homogeneous Type

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Abstract

The continuity properties of potentials on spaces of homogeneous type which are generalizations of Riesz potentials are investigated.

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1 Introduction

In the course of long years potential theory was considered as one part of mathematical physics. However, as a result of long-term two century development this theory became a vast independent field research, enriched by a whole series of new direction. At present concepts and methods of potential theory are applied not only in mathematical physics, but also in function theory, functional analysis, probability theory and harmonic analysis.

The significant applications in function theory and harmonic analysis belong to Riesz potentials. General direction of researches of Riesz potentials in function theory and harmonic analysis involve such problems as: a) studying of Riesz potentials as linear integral operators mapping from one functional space to another one including weighted spaces; b) studying of Riesz potentials as a function of the argument, namely, continuity, differentiability, behavior at infinity etc.; c) researching of generalized Riesz potentials: consideration of Riesz potentials with homogeneous kernels, changing of shift by generalized shift, changing of Euclidean distance by anisotropic kernel.

Let R^n be the n -dimensional Euclidean space and $0 < \alpha < n$. The classical Riesz potential of a locally integrable function $f : R^n \to R$ is defined by

$$I_\alpha f(x) = \int\limits_{R^n} |x-y|^{\alpha-n} f(y)\,dy,$$

where $|\cdot|$ denotes the Euclidean norm.

The classical Riesz potentials was well studied. A detail analysis and investigations of the classical Riesz potentials were given in monographs [1], [5], [8], [10]. The continuity of the potential $I_\alpha f(x)$ as a function of x was studied in [6] and [7]. In [3] results on the finiteness and continuity of the classical Riesz potentials were given in [6] were generalized for Riesz potentials with non-isotropic ketrnels depended on λ -distance.

By Sobolev imbedding theorem, it is known that if $I_\alpha f(x)$ is finite almost everywhere and if $p > \frac{n}{\alpha}$, then $I_\alpha f$ is a bounded operator mapping from $L_p(R^n)$ to $C(R^n)$. But this fact doesn't hold if $p \le \frac{n}{\alpha}$. Therefore, Y. Mizuta ([6]) found a subclass of $L_p(R^n)$ with $p = \frac{n}{\alpha}$ not contained in any class $L_q(R^n)$ different from $L_p(R^n)$, such that $I_\alpha f$ is a bounded operator mapping from this subclass to $C(R^n)$.

Using Mizuta's modified methods the continuity properties of potentials on abstract spaces which are generalizations of Riesz potentials are investigated in this paper.

2 Continuity of Potentials

Let X be a set. A function $\rho : X \times X \to [0,\infty)$ is called quasi-metric if:

1) $\rho(x,y) = 0 \;\Leftrightarrow\; x = y$
2) $\rho(x,y) = \rho(y,x)$;
3) there is a constant $c > 0$ such that for every x, y, $z \in X$

$$\rho(x,y) \le c\,(\rho(x,z) + \rho(z,y)).$$

If (X,ρ) is a set endowed with a quasi-metric, the balls

$$B(x,r) = \{y \in X : \;\rho(x,y) < r\}$$

(for $x \in X$ and $r > 0$) form a base for a complete system of neighborhoods of X is a Hausdorff space. Note that the balls are not in general open sets; if they are, then form a base for the topology of X .

Proposition 2.1 *Let X be a set together with a quasi-metric ρ and a non-negative Borel measure μ on X with $supp\mu = X$, $diam\, X = \infty$ and f be a μ-locally integrable function on X . Suppose that a function $K : (0,\infty) \to [0,\infty)$ satisfy the following conditions:*

(K_1) $K(t)$ is a almost decreasing function, i.e., there exists a constant $D > 1$ such that

$$K(s_2) \le DK(s_1), \quad for \quad 0 < s_1 < s_2 < \infty;$$

(K_2) *there exists a constant* $M \ge 1$ *such that* $K(r) \le MK(2r)$, *for* $r > 0$;
(K_3) $\int\limits_{B(x,r)} K(\rho(x,y))\, d\mu(y) < \infty$.

Then for the finiteness of the operator

$$U_K f(x) = \int\limits_X K(\rho(x,y))\, f(y)\, d\mu(y) \tag{1}$$

μ *-almost everywhere on* X *it is necessary and sufficient that one of the following equivalent conditions is fulfilled:*

(i) *there exists* $x_0 \in X$ *such that*

$$\int\limits_{X \setminus B(x_0,1)} K(\rho(x_0,y))\, f(y)\, d\mu(y) < \infty;$$

(ii) *for arbitrary* $x \in X$

$$\int\limits_{X \setminus B(x,1)} K(\rho(x,y))\, f(y)\, d\mu(y) < \infty;$$

(iii) $\int\limits_X K(1 + \rho(0,y))\, f(y)\, d\mu(y) < \infty$.

Proof First we show that it follows from (i) that integral (1) is finite μ -a.e. on X. For this purpose we write

$$\left| \int\limits_{B(x_0,1)} U_K f(x)\, d\mu(x) \right| \le \int\limits_{B(x_0,1)} d\mu(x) \left[\int\limits_{B(x_0,1+c)} K(\rho(x,y))\, |f(y)|\, d\mu(y) \right.$$

$$\left. + \int\limits_{X \setminus B(x_0,1+c)} K(\rho(x,y))\, |f(y)|\, d\mu(y) \right] = J_1 + J_2.$$

Consider J_1. If $y \in B(x_0, 1+c)$ and $x \in B(x_0, 1)$, then

$$\{y : \rho(x_0,y) < 1 + c\} \subset \{y : \rho(0,y) < c(1 + c + \rho(0,x_0))\};$$

$$\{x : \rho(x_0,x) < 1\} \subset \{x : \rho(x,y) < c(2 + c)\}.$$

By applying Fubini's theorem

$$J_1 \le \int\limits_{B(x_0,1+c)} |f(y)|\, d\mu(y) \int\limits_{B(x_0,1)} K(\rho(x,y))\, d\mu(x)$$

$$\leq \int\limits_{B(0,c(1+c+\rho(0,x_0)))} |f(y)|\, d\mu(y) \int\limits_{B(y,c(2+c))} K(\rho(x,y))\, d\mu(x) < \infty.$$

Consider J_2. If $x \in B(x_0,1)$ and $y \in X \backslash B(x_0, 1+c)$, then

$$\rho(x,y) > c^{-1}\rho(x_0,y) - 1 \geq \frac{c^{-1}}{1+c}\rho(x_0,y).$$

It is clear that there exists positive integer n such that $\frac{c^{-1}}{1+c} \geq 2^{-n}$. Then from (K_1) and (K_2)

$$J_2 \leq DM^n \int\limits_{B(x_0,1)} d\mu(x) \int\limits_{X\backslash B(x_0,1+c)} K(\rho(x_0,y))\, |f(y)|\, d\mu(y)$$

$$= DM^n \mu(B(x_0,1)) \int\limits_{X\backslash B(x_0,1+c)} K(\rho(x_0,y))\, |f(y)|\, d\mu(y).$$

It follows from (i) that $J_2 < \infty$. Therefore integral (1) is finite a.e. on X.

Now we show that $(i) \Rightarrow (ii)$. If $\rho(x,y) \geq 1$, then

$$\rho(x_0,y) \leq c(\rho(x,y) + \rho(x,x_0))$$

$$\leq c(1+\rho(x,x_0))\,\rho(x,y).$$

Let n_x is a positive integer with $c(1+\rho(x,x_0)) \leq 2^{n_x}$. Then

$$K(\rho(x,y)) \leq DK\left(2^{-n_x}\rho(x_0,y)\right) \leq DM^{n_x}K(\rho(x_0,y))$$

and

$$\int\limits_{X\backslash(x,1)} K(\rho(x,y))\, f(y)\, d\mu(y) \leq DK(1) \int\limits_{B(x_0,1)} |f(y)|\, d\mu(y)$$

$$+ \int\limits_{X\backslash B(x_0,1)\cap(X\backslash B(x,1))} K(\rho(x,y))\, f(y)\, d\mu(y)$$

$$\leq DK(1) \int\limits_{B(x_0,1)} f(y)\, d\mu(y)$$

$$+DM^{n_x} \int\limits_{X\backslash B(x_0,1)} K(\rho(x_0,y))\, f(y)\, d\mu(y).$$

Hence $(i) \Rightarrow (ii)$.

Let us show that $(i) \Leftrightarrow (iii)$. Since $\rho(x_0,y) < c(1+\rho(0,x_0))(1+\rho(0,y))$, we have

$$K(1+\rho(0,y)) \leq M_1 K(\rho(x_0,y)).$$

Then

$$\left|\int_X K(1+\rho(0,y))\, f(y)\, d\mu(y)\right| \le DK(1) \int_{B(x_0,1)} |f(y)|\, d\mu(y)$$

$$+ \int_{X\backslash B(x_0,1)} K(1+\rho(0,y))\, |f(y)|\, d\mu(y)$$

$$\le DK(1) \int_{B(x_0,1)} |f(y)|\, d\mu(y) + M_1 \int_{X\backslash B(x_0,1)} K(\rho(x_0,y))\, |f(y)|\, d\mu(y)$$

and from this it follows that $(i) \Rightarrow (iii)$. If $\rho(x_0,y) \ge 1$, then

$$1+\rho(0,y) \le \rho(x_0,y)\,(1+c(\rho(0,x_0)+1)).$$

Hence

$$\int_{X\backslash B(x_0,1)} K(\rho(x_0,y))\, |f(y)|\, d\mu(y)$$

$$\le M_2 \int_X K(1+\rho(0,y))\, |f(y)|\, d\mu(y)$$

Therefore $(iii) \Rightarrow (i)$. The proof is completed.

Let $\phi(r)$ be a strictly increasing on $(0,\infty)$ and $\lim_{r\downarrow 0} \phi(r) = 0$. A space $(X, \rho, \mu)_\phi$ is a set X together with a quasi-metric ρ and a nonnegative Borel measure μ on X with $supp\mu = X$, $diam\, X = \infty$ and there exists a constant $C \ge 1$ such that for all $r > 0$ and all $x \in X$

$$C^{-1}\phi(r) \le \mu(B(x,r)) \le C\phi(r).$$

If $\phi(2r) \le C_1\phi(r)$, then $(X, \rho, \mu)_d$ is a space of homogeneous type (see [2]).

Lemma 2.2 *Let the space $(X, \rho, \mu)_\phi$ be given, $K : (0,\infty) \to [0,\infty)$ be a continuous function satisfying the conditions (K_1), (K_2) and*

(K_4) there exist a constant $F > 0$ and $0 < \sigma < 1$ such that

$$\int_{B(x,r)} K(\rho(x,y))\; d\mu(y) < F\phi(r)^\sigma,\ \ for\ any\ r > 0.$$

Suppose that $p = \frac{1}{\sigma}$, f is a μ - locally integrable function on X satisfying the condition

$$\int_X |f(y)|^p\, w(|f(y)|)\, d\mu(y) < \infty, \tag{2}$$

where

(w_1) *w is a positive, monotone increasing function on the interval* $(0,\infty)$*;*

(w_2) $\int\limits_1^\infty w(r)^{-\frac{1}{p-1}} r^{-1}dr < \infty$*;*

(w_3) *there exists a constant* $A>0$ *such that*

$$w(2r) < Aw(r), \quad for\ any\ r>0.$$

Then there exists a positive constant L such that

$$\int\limits_{\{y\in X:|f(y)|\geq a\}} K(\rho(x,y))|f(y)|\,d\mu(y)$$

$$\leq L\left(\int\limits_{\{y\in X:|f(y)|\geq a\}} |f(y)|^p w(|f(y)|)\,d\mu(y)\right)^{\frac{1}{p}} \left(\int\limits_a^\infty w(t)^{-\frac{1}{p-1}} t^{-1}dt\right)^{\frac{1}{p'}},$$

for any $a>0$*, where* $\frac{1}{p}+\frac{1}{p'}=1$*.*

Proof For $j=1,2,...$ define

$$X_j = \left\{y\in X : 2^{j-1}a \leq |f(y)| < 2^j a\right\}.$$

Let

$$r_j = \phi^{-1}(\mu(X_j)),$$

where ϕ^{-1} is a inverse function of ϕ. Then

$$C^{-1}\mu(X_j) \leq \mu(B(0,r_j)) \leq C\mu(X_j).$$

Hence

$$\int\limits_{X_j} K(\rho(x,y))\,d\mu(y) \leq \int\limits_{B(x,r_j)} K(\rho(x,y))\,d\mu(y) + \int\limits_{X_j\backslash B(x,r_j)} K(\rho(x,y))\,d\mu(y)$$

$$\leq \int\limits_{B(x,r_j)} K(\rho(x,y))\,d\mu(y) + DK(r_j)\int\limits_{X_j\backslash B(x,r_j)} d\mu(y)$$

$$\leq \int\limits_{B(x,r_j)} K(\rho(x,y))\,d\mu(y) + DCK(r_j)\,\mu(B(x,r_j))$$

$$\leq \left(1+D^2C\right)\int\limits_{B(x,r_j)} K(\rho(x,y))\,d\mu(y) \leq M_1\phi(r_j)^\sigma,$$

where $M_1 = (1+D^2C)\,F$. Therefore

$$\int\limits_{\{y\in X:|f(y)|\geq a\}} K(\rho(x,y))|f(y)|\,d\mu(y) = \int\limits_{X_j}\sum_{j=1}^\infty K(\rho(x,y))|f(y)|\,d\mu(y)$$

$$\leq \sum_{j=1}^{\infty} 2^j a \int_{X_j} K(\rho(x,y))\, d\mu(y) \leq M_1 \sum_{j=1}^{\infty} 2^j a \phi(r_j)^{\sigma}$$

$$= 2M_1 \sum_{j=1}^{\infty} 2^{j-1} a w\left(2^j a\right)^{\frac{1}{p}} \left(\mu(X_j)\right)^{\frac{1}{p}} w\left(2^j a\right)^{-\frac{1}{p}}$$

$$\leq 2M_1 A^{\frac{1}{p}} \sum_{j=1}^{\infty} 2^{j-1} a w\left(2^{j-1} a\right)^{\frac{1}{p}} \left(\mu(X_j)\right)^{\frac{1}{p}} w\left(2^j a\right)^{-\frac{1}{p}}$$

$$\leq 2M_1 A^{\frac{1}{p}} \left[\sum_{j=1}^{\infty} \left(2^{j-1} a\right)^p w\left(2^{j-1} a\right) \mu(X_j)\right]^{\frac{1}{p}} \left[\sum_{j=1}^{\infty} w\left(2^j a\right)^{-\frac{1}{p-1}}\right]^{\frac{1}{p'}}$$

$$\leq 2M_1 A^{\frac{1}{p}} \left(\int_{\{y\in X:|f(y)|\geq a\}} |f(y)|^p w(|f(y)|)\, d\mu(y)\right)^{\frac{1}{p}} \left(\int_a^{\infty} w(t)^{-\frac{1}{p-1}} t^{-1} dt\right)^{\frac{1}{p'}}.$$

The lemma is proved.

Theorem 2.3 *Let the conditions of Lemma 2.2 be satisfied and f satisfy the condition (iii). Then $U_K f(x)$ is continuous on X.*

Proof For $x \in G$ and $r \in (0,\infty)$, we can write

$$U_K f(x) = \int_{B(x_0,r)} K(\rho(x,y)) f(y)\, d\mu(y)$$

$$+ \int_{X\setminus B(x_0,r)} K(\rho(x,y)) f(y)\, d\mu(y) = P_r(x) + Q_r(x)$$

and

$$|P_r(x)| \leq \int_{B(x_0,r)} K(\rho(x,y))\, d\mu(y)$$

$$+ \int_{B(x_0,r)\cap\{y\in X:|f(y)|>1\}} K(\rho(x,y))\, |f(y)|\, d\mu(y) = P_r'(x) + P_r''(x). \tag{3}$$

Then

$$P_r'(x) \leq \int_{B(x,r)} K(\rho(x,y))\, d\mu(y) + \int_{B(x_0,r)\cap(X\setminus B(x,r))} K(\rho(x,y))\, d\mu(y)$$

$$\leq \int_{B(x,r)} K(\rho(x,y))\, d\mu(y) + DK(r) \int_{B(x_0,r)} d\mu(y)$$

$$\leq \int_{B(x,r)} K(\rho(x,y))\, d\mu(y) + DCK(r)\,\phi(r)$$

$$\leq \int\limits_{B(x,r)} K\left(\rho\left(x,y\right)\right)d\mu\left(y\right)+DC^{2}K\left(r\right)\int\limits_{B(x,r)} d\mu\left(y\right)$$

$$\leq \left(1+D^{2}C^{2}\right)\int\limits_{B(x,r)} K\left(\rho\left(x,y\right)\right)d\mu\left(y\right)\leq \left(1+D^{2}C^{2}\right)F\phi\left(r\right)^{\sigma}. \qquad (4)$$

By Lemma 2.2

$$P_{r}''\left(x\right)\leq L\left(\int\limits_{\{y\in X:|f(y)|\geq 1\}\cap B(x_{0},r)} \left|f\left(y\right)\right|^{p}w\left(\left|f\left(y\right)\right|\right)d\mu\left(y\right)\right)^{\frac{1}{p}}\times$$

$$\times\left(\int\limits_{1}^{\infty} w\left(t\right)^{-\frac{1}{p-1}}t^{-1}dt\right)^{\frac{1}{p'}} \qquad (5)$$

Then

$$\left|P_{r}\left(x\right)\right|<\infty.$$

By conditions of the present theorem

$$\left|Q_{r}\left(x\right)\right|<\infty.$$

Therefore

$$\left|U_{K}f\left(x\right)\right|<\infty.$$

Let $r=2c\rho\left(x,x_{0}\right)$. Then by (3), (4) and (5)

$$P_{r}\left(x\right)\rightarrow 0,\text{as }\rho\left(x,x_{0}\right)\rightarrow 0.$$

If $\rho\left(x_{0},y\right)>r$, then

$$\rho\left(x,y\right)\geq c^{-1}\rho\left(x_{0},y\right)-\rho\left(x,x_{0}\right)\geq$$

$$\geq c^{-1}\rho\left(x_{0},y\right)-\frac{c^{-1}}{2}\rho\left(x_{0},y\right)=\frac{c^{-1}}{2}\rho\left(x_{0},y\right).$$

It is clear that there exists natural number n such that $\frac{c^{-1}}{2}\geq 2^{-n}$. Hence

$$K\left(\rho\left(x,y\right)\right)\leq DK\left(\frac{c^{-1}}{2}\rho\left(x_{0},y\right)\right)\leq DM^{n}K\left(\rho\left(x_{0},y\right)\right).$$

Define

$$f_{r}\left(x\right)=\begin{cases} f\left(x\right),\text{if } x\in X\backslash B\left(x_{0},r\right)\\ 0,\ \text{if } x\in B(x_{0},r)\end{cases}$$

Then

$$Q_{r}\left(x\right)=\int\limits_{X} K\left(\rho\left(x,y\right)\right)f_{r}\left(y\right)d\mu\left(y\right).$$

Further

$$K(\rho(x,y))\,|f_r(y)| \le DM^d K(\rho(x_0,y))\,|f(y)|\,.$$

Since $K(r)$ is continues, we have

$$K(\rho(x,y))\,|f_r(y)| \to K(\rho(x_0,y))\,|f(y)|\,, \text{ as } \rho(x,x_0)\to 0.$$

By applying Lebesgue dominated convergence theorem we obtain

$$\lim_{\rho(x,x_0)\to 0} U_K f(x) = \lim_{\rho(x,x_0)\to 0} Q_r(x) = U_K f(x_0)\,.$$

The proof is completed.

Remark 2.4 *As examples of $w(r)$ satisfying (w_1), (w_2), (w_3) may serve*

$$w(r) = \log(2+r)^{\delta}\,;$$

$$w(r) = \log(2+r)^{p-1}\log(2+\log(2+r))^{\delta}$$

and so on, where $\delta > p-1 > 0$.

Remark 2.5 *Let $\phi(r) = r^d$. Then the functions $K(t) = t^{\alpha-d}$ and $K(t) = t^{\alpha-d}\log(1+t^{-1})$, for $0<\alpha<d$ satisfy conditions (K_1), (K_2), (K_4). We prove (K_4) for $K(t) = t^{\alpha-d}\log(1+t^{-1})$. Let $\beta\in(0,\alpha)$. Then*

$$\int\limits_{B(x,r)} \rho(x,y)^{\alpha-d}\log\left(1+\rho(x,y)^{-1}\right)d\mu(y)$$

$$\le \int\limits_{B(x,r)} \rho(x,y)^{\alpha-d-\beta}\,d\mu(y)$$

$$= \sum_{j=0}^{\infty}\ \int\limits_{2^{-j-1}r\le\rho(x,y)<2^{-j}r} \rho(x,y)^{\alpha-d-\beta}\,d\mu(y)$$

$$\le \sum_{j=0}^{\infty}\left(2^{-j-1}r\right)^{\alpha-d-\beta} \int\limits_{2^{-j-1}r\le\rho(x,y)<2^{-j}r} d\mu(y)$$

$$\le C\sum_{j=0}^{\infty}\left(2^{-j-1}r\right)^{\alpha-d-\beta}\left(2^{-j}r\right)^{d} \le C_1 r^{\alpha-\beta} = C_1\phi(r)^{\sigma}\,,$$

where $\sigma = \frac{\alpha-\beta}{d}$.

It is easy to see that $\sigma = \frac{\alpha}{d}$ if $K(r) = r^{\alpha-d}$.

3 Finite Differences of Riesz Potentials

Let $d > 0$ and $0 < \theta \leq 1$. A space of homogeneous type $(X, \rho, \mu)_{d,\theta}$ (see [4], [9]) is a set X together with a quasi-metric ρ and a nonnegative Borel measure μ on X with $supp\mu = X$, $diam X = \infty$ and there exists a constant $C > 0$ such that for all $r > 0$ and all $x, y, z \in X$

$$C^{-1}r^d < \mu\left(B\left(x, r\right)\right) < Cr^d$$

and

$$\left|\rho\left(x, y\right) - \rho\left(z, y\right)\right| \leq C_0 \rho\left(x, z\right)^{\theta}\left[\rho\left(x, y\right) + \rho\left(z, y\right)\right]^{1-\theta}. \tag{6}$$

It is easy to see that if ρ is a metric, then ρ satisfies (9) for all $\theta \in (0, 1]$.

Let $(X, \rho, \mu)_{d,\theta}$ is a space of homogeneous type and $0 < \alpha < d$. Consider the Riesz potential

$$R_\alpha f\left(x\right) = \int\limits_X \rho\left(x, y\right)^{\alpha-d} f\left(y\right) d\mu\left(y\right).$$

Lemma 3.1 *([7], [11]). Let $w(r)$ be a function satisfying (w_1) and (w_4) there exists $A_1 > 0$ such that*

$$A_1^{-1} w\left(r\right) \leq w\left(r^2\right) \leq A_1 w\left(r\right).$$

Then for $\gamma > 0$

$$A_\gamma^{-1} w\left(r\right) \leq w\left(r^\gamma\right) \leq A_\gamma w\left(r\right)$$

and

$$\frac{s_1^\gamma}{w\left(s_1\right)} \leq A_2 \frac{s_2^\gamma}{w\left(s_2\right)} \text{ whenever } s_2 > s_1 > 0.$$

Theorem 3.2 *Let $(X, \rho, \mu)_{d,\theta}$ be a space of homogeneous type, $p = \frac{d}{\alpha} > 0$, $w(r)$ be a function satisfying (w_1), (w_2), (w_4), $f(x)$ be a μ -locally integrable function on X satisfying (2) and*

$$\int\limits_X \left(1 + \rho\left(0, y\right)\right)^{\alpha-d} \left|f\left(y\right)\right| d\mu\left(y\right) < \infty.$$

Then

$$\left|R_\alpha f\left(x\right) - R_\alpha f\left(z\right)\right| = o\left(w^*\left(\rho\left(x, z\right)\right)\right), \text{ as } \rho\left(x, z\right) \to 0,$$

where

$$w^*\left(r\right) = \left(\int\limits_0^r w\left(t^{-1}\right)^{-\frac{1}{p-1}} t^{-1} dt\right)^{1-\frac{1}{p}}.$$

Proof Let $r = \rho(x, z) < \frac{1}{2}$. Then we can write

$$|R_\alpha f(x) - R_\alpha f(z)| \le \int_{B(x,2cr)} \rho(z, y)^{\alpha-d} |f(y)| \, d\mu(y)$$

$$+ \int_{B(x,2cr)} \rho(x, y)^{\alpha-d} |f(y)| \, d\mu(y)$$

$$+ \int_{X\setminus B(x,2cr)} \left|\rho(x, y)^{\alpha-d} - \rho(z, y)^{\alpha-d}\right| |f(y)| \, d\mu(y) = J_1(z) + J_2(x) + J_3(x, z).$$

Consider $J_1(z)$. If $y \in B(x, 2cr)$, then

$$\rho(y, z) \le c(\rho(x, y) + \rho(x, z)) \le c_1 r,$$

where $c_1 = c(2c + 1)$. Let $\alpha - \theta < \gamma < \alpha$. By Lemma 2.2 and Lemma 3.1

$$J_1(z) \le \int_{B(z,c_1r)} \rho(z, y)^{\alpha-d-\gamma} d\mu(y) + \int_{\left\{y: B(z,c_1r), |f(y)| > \rho(z,y)^{-\gamma}\right\}} \rho(z, y)^{\alpha-d} |f(y)| \, d\mu(y)$$

$$\le \sum_{j=0}^{\infty} \left(2^{-j-1} c_1 r\right)^{\alpha-d-\gamma} \int_{2^{-j-1}c_1r \le \rho(z,y) < 2^{-j}c_1r} d\mu(y)$$

$$+L \left(\int_{\left\{y: B(z,c_1r)\, |f(y)| > \rho(z,y)^{-\gamma}\right\}} |f(y)|^p w(|f(y)|) \, d\mu(y) \right)^{\frac{1}{p}} \left(\int_{(c_1r)^{-\gamma}}^{\infty} w(t)^{-\frac{1}{p-1}} t^{-1} dt \right)^{\frac{1}{p'}}$$

$$\le C_1 r^{\alpha-\gamma} + L \left(\int_{B(z,c_1r)} |f(y)|^p w(|f(y)|) \, d\mu(y) \right)^{\frac{1}{p}} \left(\gamma \int_0^r w\left((c_1 t)^{-\gamma}\right)^{-\frac{1}{p-1}} t^{-1} dt \right)^{\frac{1}{p'}}$$

$$\le C_1 r^{\alpha-\gamma} + C_2 \left(\int_{B(z,c_1r)} |f(y)|^p w(|f(y)|) \, d\mu(y) \right)^{\frac{1}{p}} w^*(r).$$

If $\rho(z, y) < c_1 r$, then $\rho(x, y) < c_2 r$, where $c_2 = c(c_1 + 1)$. Thus

$$J_1(z) \le C_1 r^{\alpha-\gamma} + C_2 \left(\int_{B(x,c_2r)} |f(y)|^p w(|f(y)|) \, d\mu(y) \right)^{\frac{1}{p}} w^*(r).$$

By the same way

$$J_2(x) \le C_1 r^{\alpha-\gamma} + C_2 \left(\int_{B(x,c_2r)} |f(y)|^p w(|f(y)|) \, d\mu(y) \right)^{\frac{1}{p}} w^*(r).$$

Let $\overline{u} = \max\left(\rho\left(x,y\right),\rho\left(z,y\right)\right)$ and $\underline{u} = \min\left(\rho\left(x,y\right),\rho\left(z,y\right)\right)$. By mean value theorem, there exists $u_0 \in \left(\underline{u},\overline{u}\right)$ such that

$$\left|\rho\left(x,y\right)^{\alpha-d} - \rho\left(z,y\right)^{\alpha-d}\right| = \underline{u}^{\alpha-d} - \overline{u}^{\alpha-d} = \left(d-\alpha\right)\left(\overline{u}-\underline{u}\right)u_0^{\alpha-d-1}.$$

By (6) we can write

$$\overline{u}-\underline{u} \le C_0\rho\left(x,z\right)^{\theta}\left[\rho\left(x,y\right)+\rho\left(z,y\right)\right]^{1-\theta} \le C_3 r^{\theta}\rho\left(x,y\right)^{1-\theta} \text{ whenever } \rho\left(x,y\right) \ge 2cr.$$

Then

$$\left|\rho\left(x,y\right)^{\alpha-d} - \rho\left(z,y\right)^{\alpha-d}\right| \le C_4 r^{\theta}\rho\left(x,y\right)^{\alpha-d-\theta} \text{ whenever } \rho\left(x,y\right) \ge 2cr.$$

So that

$$J_3\left(x,z\right) \le C_4 r^{\theta}\int\limits_{X\backslash B(x,2cr)}\rho\left(x,y\right)^{\alpha-d-\theta}\left|f\left(y\right)\right|d\mu\left(y\right)$$

$$\le C_4 r^{\theta}\int\limits_{X\backslash B(x,2cr)}\rho\left(x,y\right)^{\alpha-d-\theta-\gamma}d\mu\left(y\right)$$

$$+C_4 r^{\theta} w\left(r^{-\gamma}\right)^{-\frac{1}{p}}\int\limits_{\{y:X\backslash B(x,2cr),|f(y)|>r^{-\gamma}\}}\rho\left(x,y\right)^{\alpha-d-\theta}\left[\left|f\left(y\right)\right|w\left(\left|f\left(y\right)\right|\right)^{\frac{1}{p}}\right]d\mu\left(y\right)$$

$$\le C_4 C r^{\theta}\left(2cr\right)^{\alpha-\theta-\gamma} + C_4 r^{\theta} w\left(r^{-\gamma}\right)^{-\frac{1}{p}}\left(\int\limits_{X\backslash B(x,2cr)}\rho\left(x,y\right)^{(\alpha-d-\theta)p'}d\mu\left(y\right)\right)^{\frac{1}{p'}}$$

$$\times\left(\int\limits_{X\backslash B(x,2cr)}\left|f\left(y\right)\right|^{p} w\left(\left|f\left(y\right)\right|\right)d\mu\left(y\right)\right)^{\frac{1}{p}}$$

$$\le C_5 r^{\alpha-\gamma} + C_6 r^{\theta} w\left(r^{-1}\right)^{-\frac{1}{p}}\left(\sum_{j=0}^{\infty}\int\limits_{2^{j+1}cr\le\rho(x,y)<2^{j+2}cr}\rho\left(x,y\right)^{-d-\theta p'}d\mu\left(y\right)\right)^{\frac{1}{p'}}$$

$$\le C_5 r^{\alpha-\gamma} + C_6 r^{\theta} w\left(r^{-1}\right)^{-\frac{1}{p}}\left(\sum_{j=0}^{\infty}C\left(2^{j+1}cr\right)^{-d-\theta p'}\left(2^{j+2}cr\right)^{d}\right)^{\frac{1}{p'}}$$

$$\le C_5 r^{\alpha-\gamma} + C_7 w\left(r^{-1}\right)^{-\frac{1}{p}}.$$

By (w_2) we see that

$$w^{*}\left(r\right) \ge \left(\int\limits_{r^2}^{r} w\left(t^{-1}\right)^{-\frac{1}{p-1}} t^{-1}dt\right)^{\frac{1}{p'}} \ge C_8 w\left(r^{-1}\right)^{-\frac{1}{p}}\log\left(r^{-1}\right)^{\frac{1}{p'}}. \qquad (7)$$

By Lemma 3.1

$$s^{\alpha-d} \leq C_9 w\left(s^{-1}\right)^{-1} \tag{8}$$

for $0 < s < 1$.

Thus we establish

$$J_3(x,z) \leq C_{10} w^*(r) \log(r^{-1})^{-\frac{1}{p'}}.$$

Now it follows that

$$|R_\alpha f(x) - R_\alpha f(z)| \leq 2C_1 r^{\alpha-\gamma}$$

$$+2C_2 \left(\int_{B(x,c_2 r)} |f(y)|^p w(|f(y)|)\, d\mu(y) \right)^{\frac{1}{p}} w^*(r) + C_{10} w^*(r) \log(r^{-1})^{-\frac{1}{p'}}$$

and using (7) and (8) we have the required result.

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