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Best Simultaneous Approximation in Function Spaces

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Abstract

Let X be a Banach space and G be a closed subspace of X. We say G is $2-$simultaneously proximinal in X if for any x_1, x_2 in X, there exists some $y \in G$ such that $\|x_1 - y\| + \|x_2 - y\| = \inf\{\|x_1 - z\| + \|x_2 - z\| : z \in G\} = d(\{x_1, x_2\}, G)$. In this paper, we give a formula for $d(\{x_1, x_2\}, G)$ in vector valued integrable functions. Results on simultaneous proximinality in such spaces will be presented.

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I. Introduction. Let X be a Banach space and G be a closed subspace of X. For $E \subset X$, we write

$d_1(E, G) = \inf\{\sum_{e \in E} \|e - y\| : y \in G\}$. **Such infimum need not be attained. In case the infimum is attained for any subset $E \subset X$, we say that G is $|E|-$simultaneously proximinal in X, where $|E|$ is the cardinality of E. We say G is $2-$simultaneously proximinal in X if for any x_1, x_2 in X, there exists some $y \in G$ such that $\|x_1 - y\| + \|x_2 - y\| = \inf\{\|x_1 - z\| + \|x_2 - z\| : z \in G\} = d(\{x_1, x_2\}, G)$. In case $|E| = 1$ then $1-$simultaneous proximinality is just proximinality .The first result on $2-$simultaneous approximation in $C(I, R)$, the space of continuous real valued functions on some compact interval I, is due to Dunham [2]. Many good results had appeared since then. We refer to [1], [4], [5], [6], [7], [8], and [9]. However, all these results except for [7], dealt with the space of continuous functions with d_∞ instead of d_1.**

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It is the object of this paper to study 2−simultaneous approximation in vector valued function spaces with d_1 distance. We present a formula for $d_1(E,G)$ when X is the space of Bochner integrable functions on some interval I. Many other results are presented.

Let I be a compact interval. With no loss of generality we assume $I=[0,1]$. For a Banach space X, $L^1(I,X)$ denotes the space of strongly measurable functions f on I such that $\int \|f(t)\|\,dt<\infty$. For $f_1,f_2\in L^1(I,X)$ and G a closed subspace of X,we set

$\mathbf{d}_1(\{\mathbf{f}_1,\mathbf{f}_2\})=\inf\left(\int\left(\|f_1(t)-g(t)\|+\|f_2(t)-g(t)\|\right)\mathbf{dt}:\mathbf{g}\in\mathbf{L}^1(\mathbf{I},\mathbf{G})\right).$

For any Banach space Y and closed subspace F of Y, we set:

$J(Y)=Y\underset{1}{\oplus}Y$ **with** $\|x+y\|=\|x\|+\|y\|$

$D(F)=\{(z,z):z\in F\}$, **with** $\|(z,z)\|=\|z\|+\|z\|$.

Clearly, $D(F)$ is a closed subspace of $J(Y)$. Further, F is $2-$simultaneously proximinal in Y if and only if $D(F)$ is proximinal in $J(Y)$.

II. <u>The Distance Formula.</u>

In this section we deduce a distance formula for best simultaneous approximation in $L^1(I,X)$. We discuss only the distance for two functions.

Theorem 2.1. **Let f_1, $f_2\in L^1(I,X)$, and G be a closed subspace of X. Define**

$$\varphi(s)=\inf\{\|f_1-z\|+\|f_2-z\|:z\in G\}.$$

Then φ is measurable and

$$\int\varphi(s)ds=\inf\{\|f_1-h\|+\|f_2-h\|:h\in L^1(I,G)\}=d(\{h_1,h_2\},L^1(I,G)).$$

Proof. **Let f_1,f_2 be any two elements in $L^1(I,X)$. Then there exist two sequences (f_{1n}) and (f_{2n}),of simple functions, such that $\|f_1(t)-f_{1n}(t)\|\longrightarrow 0$ and $\|f_2(t)-f_{2n}(t)\|\longrightarrow 0$. Now, the function**

$d((x,y),D(G))=\inf\{\|x-z\|+\|y-z\|:z\in G\}$ **is continuous. Consequently $\lim(d(f_{1n}(t),f_{2n}(t)),D(G))=d((f_1(t),f_2(t)),D(G))$. Define the sequence of functions $\varphi_n:I\longrightarrow R$, $\boldsymbol{\varphi}_n(\mathbf{s})=\mathbf{d}((\mathbf{f}_{1n}(\mathbf{s}),\mathbf{f}_{2n}(\mathbf{s})),\mathbf{D}(\mathbf{G}))=$ $\inf\{\|f_{1n}(s)-g\|+\|f_{2n}(s)-g\|:g\in D(G)\}$.Since f_{1n} and f_{2n} are simple functions, then we can assume: $f_{1n}=\sum_{i=1}^{n}1_{A_i}\otimes x_i$ and $f_{2n}=\sum_{i=1}^{n}1_{A_i}\otimes y_i$ with A_i are disjoint. But then one can see that**

$$\boldsymbol{\varphi}_n(\mathbf{s})=\inf\left\{\sum\mathbf{1}_{A_i}(\mathbf{s})\left(\|x_i-g\|+\|y_i-g\|\right)\right)$$
$$=\sum 1_{A_i}(s)\inf(\|x_i-g\|+\|y_i-g\|),\text{ since the } A_i^{,s} \text{ are disjoined.}$$

Hence, each φ_n is a simple function, and consequently the function φ is measurable.

Now, let $g\in L^1(I,G)$. Then

$$\|f_1-g\|+\|f_2-g\|=\int_I\left(\|f_1(s)-g(s)\|+\|f_2(s)-g(s)\|\right)\mathbf{ds}$$

$$\geq \int_I d((f_{1n}(s), f_{2n}(s)), D(G))ds = \|\varphi\|_1 .$$

Taking the infimum over all $g \in L^1(I, G)$, we get

$d((f_1, f_2), L^1(I, G)) \geq \|\varphi\|_1$**..(1).**

For the reverse inequality: Let $\epsilon > 0$ be arbitrary. Choose P and Q, simple functions such that $P = \sum_{i=1}^{m} 1_{B_i} \otimes z_i$, $Q = \sum_{i=1}^{m} 1_{B_i} \otimes w_i$, $\|f_1 - P\| < \epsilon$, and $\|f_2 - Q\| < \epsilon$, where the $B_i^{;s}$ are disjoint and $\mu(B_i) > 0$ for all i.

From the definition of the distance there exists $h_i \in G$ such that $\|z_i - h_i\| + \|w_i - h_i\| < d((z_i, w_i), D(G)) + \epsilon$. Now:

$$\mathbf{d}((\mathbf{f}_1, \mathbf{f}_2), \mathbf{D}(\mathbf{L}^1(\mathbf{I}, \mathbf{G}))) = \inf\{ \|f_1 - h\| + \|f_2 - h\| : \mathbf{h} \in \mathbf{L}^1(\mathbf{I}, \mathbf{G})\}$$

$$\leq \inf\{\|f_1 - P\| + \|f_2 - Q\| + \|h - P\| + \|h - Q\| : h \in L^1(I, G)\}$$

$$\leq 2\epsilon + \inf\{\|h - P\| + \|h - Q\| : h \in L^1(I, G)\}$$

$$\leq 2\epsilon + \inf\{\|g - P\| + \|g - Q\| : g \in L^1(I, G), \text{ with } g \text{ of the form } g = \sum_{i=1}^{m} 1_{B_i} \otimes h_i\}$$

$$\leq 2\epsilon + \inf \sum_{i=1}^{m} \mu(B_i)(\|z_i - h_i\| + \|w_i - h_i\|)$$

$$\leq 2\epsilon + \inf \sum_{i=1}^{m} \mu(B_i)[d((z_i, w_i), D(G)) + \epsilon]$$

$$\leq 3\epsilon + \sum_{i=1}^{m} \int_{B_i} d((z_i, w_i), D(G))ds \quad \textbf{(since } \sum \mu(A_i) = 1\textbf{)}$$

$$= 3\epsilon \int_I d((P(s), Q(s)), D(G))ds$$

$$= 3\epsilon + \int_I \inf\{\|P(s) - z\| + \|Q(s) - z\| : z \in G\}ds$$

$$\leq 3\epsilon + \int_I \inf\{\|f_1(s) - z\| + \|f_2(s) - z\| + \|f_1(s) - P(s)\| + \|f_2(s) - Q(s)\| : z \in G\}ds$$

$$\leq 3\epsilon + \int_I \inf\{\|f_1(s) - z\| + \|f_2(s) - z\| : z \in G\}ds + (\|f_1 - P\| + \|f_2 - Q\|)$$

$$\leq 5\epsilon + \int_I \inf\{\|f_1(s) - h(s)\| + \|f_2(s) - h(s)\| : h \in L^1(I, G)\}ds$$

$\leq 5\epsilon + \|\varphi\|_1$**..(2).**

Since ϵ was arbitrary, equations (1) and (2) ends the proof.

As an application to Theorem 2.1 we have:

Theorem 2.2. **Let G be a closed subspace of the Banach space X, and $f_1, f_2 \in L^1(I, X)$. Then for any $g \in L^1(I, G)$, the following are equivalent.**

(i) g **is a best simultaneous approximant for** f_1, f_2 **in** $L^1(I,G)$.

(ii) $g(t)$ **is a best simultaneous approximant for** $f_1(t), f_2(t)$ **in** G.

Another nice application of Theorem 2.1 is

Theorem 2.3. Let G **be a closed subspace of** X. **If** $L^1(I,G)$ **is simultaneously proximinal in** $L^1(I,X)$**, then** G **is simultaneously proximinal in** X.

Proof. **Let** $x, y \in X$. **Define** $f_1 = 1 \otimes x$ **and** $f_2 = 1 \otimes y$**, where** 1 **is the constant function** 1. **Clearly** f_1 **and** f_2 **are in** $L^1(I,X)$. **By assumption, there exists** $g \in L^1(I,G)$ **such that**

$\|f_1 - g\| + \|f_2 - g\| \le \|f_1 - h\| + \|f_2 - h\|$ **for all** $h \in L^1(I,G)$.

Theorem 2.1,

$\|f_1(t) - g(t)\| + \|f_2(t) - g(t)\| \le \|f_1(t) - h(t)\| + \|f_2(t) - h(t)\|$ **for all** $h \in L^1(I,G)$. **Thus**

$\|x - g(t)\| + \|x - g(t)\| \le \|x - h(t)\| + \|x - h(t)\|$ **for all** $h \in L^1(I,G)$. **Let** h **runs over all functions of the form** $1 \otimes z$, **for** $z \in G$**, the result follows.**

Now, we give a very simple proof of one of the main results in [7].

Theorem 2.4. **Let** G **be a reflexive subspace of the Banach space** X. **Then** $L^1(I,G)$ **is simultaneously proximinal in** $L^1(I,X)$.

Proof. **Since** G **is reflexive then** $D(G)$ **is a reflexive subspace of** $G \underset{1}{\oplus} G \subseteq X \underset{1}{\oplus} X$. **Now** $L^1(I,X) \underset{1}{\oplus} L^1(I,X)$ **is isometrically isomorphic to** $L^1(I, X \underset{1}{\oplus} X)$, **and** $D(L^1(I,G))$ **is isometrically isomorphic to** $L^1(I, D(G))$. **The result now follows from the main result in [3]. That ends the proof.**

III. Further Results.

A closed subspace G **is called 1-summand in** X **if there exists a subspace(closed)** Y **such that** $X = G \underset{1}{\oplus} Y$. **It is known that [3], that a 1-summand subspace** G **of** X **is proximinal, and** $L^1(I,G)$ **is proximinal in** $L^1(I,X)$. **Now we prove:**

Theorem 3.1. **Let** G **be a 1-summand subspace of** X. **Then** G **is simultaneously proximinal.**

Proof. **Let** x, y **be any two elements in** X. **Since** X **is 1-summand, then** $X = G \underset{1}{\oplus} M$. **So** $x = x_1 + x_2$, **and** $y = y_1 + y_2$. **Let** $z = \frac{x_1 + y_1}{2}$. **Then** $x - z = \frac{x_1 - y_1}{2} + x_2$, **and** $y - z = \frac{y_1 - x_1}{2} + y_2$. **Hence**

$$\begin{aligned} \|x - z\| + \|y - z\| &= \left\|\tfrac{x_1 - y_1}{2}\right\| + \|x_2\| + \left\|\tfrac{y_1 - x_1}{2}\right\| + \|y_2\| \\ &= \|x_2\| + \|y_2\| + \|x_1 - y_1\| \\ &\le \|x_2\| + \|y_2\| + \|x_1 - w\| + \|y_1 - w\|, \text{(for any } w \in G) \end{aligned}$$

$$= (\|x_2\| + \|x_1 - w\|) + (\|y_2\| + \|y_1 - w\|)$$
$$= \|x - w\| + \|y - w\| .$$

Hence z is a best simultaneous approximation in G for x and y, and G is simultaneously proximinal.

As a corollary, we get the following:

Theorem 3.2. **If G is 1-summand in X, then $L^1(I,G)$ is simultaneously proximinal in $L^1(I,X)$.**

Proof. $L^1(I,X) = L^1(I,G \underset{1}{\oplus} M) = L^1(I,G) \underset{1}{\oplus} L^1(I,M)$. **Hence** $L^1(I,G)$ **is 1-summand in** $L^1(I,X)$. **Hence by Theorem 3.1,** $L^1(I,G)$ **is simultaneous proximinal.**

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