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On Bilateral Generating Relation for a Sequence of Functions

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Abstract

In this paper, we obtained a bilateral generating relation for a sequence of function i.e. $\{V_n^{(\alpha,\beta)}(x;a,k,s)/n = 0,1,2,\cdots\}$ which is recently introduced by Shukla and Prajapati [3] defined as (1.1) by applying Srivastava's Theorem [6], and its special cases have also been discussed.

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1 Introduction and Preliminaries

Recently, Shukla and Prajapati [3] introduced a sequence of function $\{V_n^{(\alpha,\beta)}(x;a,k,s)\}$ defined as:

$$V_n^{(\alpha,\beta)}(x;a,k,s) = \frac{1}{n!}\, x^{-\beta}\, E_\alpha\{p_k(x)\}\theta^n\Big[x^\beta E_\alpha\{-p_k(x)\}\Big] \tag{1.1}$$

where $p_k(x)$ is a polynomial in x of degree k and the differential operator $\theta \equiv x^a(s + xD)$, $D \equiv \dfrac{d}{dx}$, a and s being constants.

$E_\alpha(z)$ is Mittag-Leffler function [1] defined as:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1.2}$$

where z is a complex variable and $\Gamma(s)$ is a gamma function, $\alpha > 0$.

This sequence of function (1.1) incorporates several polynomial systems.

A general class of polynomials $G_n^{(\alpha,\beta)}(x,r,p,k)$ defined by Shukla and Prajapati [4] as:

$$G_n^{(\alpha,\beta)}(x,r,p,k) \;=\; \frac{1}{n!}\, x^{-\beta-kn}\, E_\alpha(px^r)\theta^n \Big[x^\beta E_\alpha(-px^r) \Big] \tag{1.3}$$

where $E_\alpha(z)$is Mittag-Leffler function which is defined as (1.2).

The polynomials $G_n^{(\alpha,\beta)}(x,r,p,k)$ provide an elegant unification of the various known generalizations of the classical Hermite, Laguerre and Konhauser polynomials. This is easy to prove following relation by using (1.1) and (1.3) as:

$$G_n^{(\alpha,\beta)}(x,p,k,a) \;=\; x^{-an}\, V_n^{(\alpha,\beta)}(x;a,k,0) \quad \text{(when } p_k(x) = px^k\text{)}. \tag{1.4}$$

Shukla and Prajapati [3] have also proved following linear generating relation:

$$\sum_{n=0}^{\infty} \binom{n+m}{n} V_{n+m}^{(\alpha,\beta)}(x;a,k,s)t^n \;=\; (1-ax^\alpha t)^{-\left(\frac{\beta+s}{a}\right)}$$

$$\times\, \frac{E_\alpha\{p_k(x)\}}{E_\alpha[p_k\{x(1-ax^a t)^{-1/a}\}]}\, V_n^{(\alpha,\beta)}\{x(1-ax^a t)^{-1/a};a,k,s\} \tag{1.5}$$

where $m = 0, 1, 2, \cdots$.

Srivastava [6] introduced a general method of obtaining bilinear, bilateral or mixed multilateral generating relations for all function $S_\nu^*(x)$ satisfying

$$\sum_{n=0}^{\infty} A_{\nu,n}^* S_{\nu+n}^*(x)t^n \;=\; f^*(x,t)\{g(x,t)\}^{-\nu}\, S_\nu^*(h(x,t)) \tag{1.6}$$

where ν is an arbitrary complex number. He has also proved Theorem 3, in which he showed nicely how these two general theorems are applicable to derive a large variety of bilinear, bilateral or mixed multilateral generating relations of several variables.

Theorem 2 (Srivastava [6]):

$$\text{If} \qquad F_{q,\nu}^{p,\mu}[x;y_1,\cdots,y_s;t] \;=\; \sum_{n=0}^{\infty} C_n^{\mu,\nu}\, S_{\nu+qn}^*(x)\, \Omega_{\mu+pn}(y_1,\cdots,y_s)t^n \tag{1.7}$$

and the functions $S^*_\mu(x)$ defined by (1.6), where $C^{\mu,\nu}_n \neq 0$, μ and ν are arbitrary complex numbers, p and q are positive integers, $\Omega_\mu(y_1, \cdots, y_s)$ is a non-vanishing function of s variables i.e. $y_1, \cdots, y_s$, $s \geq 1$. Then

$$\sum_{n=0}^{\infty} S^*_{\nu+n}(x) W^{p,\mu}_{n,q,\nu}(y_1, \cdots, y_s; z) t^n$$

$$= f^*(x,t)\{g(x,t)\}^{-\nu} F^{p,\mu}_{q,\nu}\Big[h(x,t); y_1, \cdots, y_s; z\Big\{\frac{t}{g(x,t)}\Big\}^q\Big] \tag{1.8}$$

where $W^{p,\mu}_{n,q,\nu}(y_1, \cdots, y_s; z)$ is a polynomial of degree $\left[\dfrac{n}{q}\right]$ in z, which is defined as:

$$W^{p,\mu}_{n,q,\nu}(y_1, \cdots, y_s; z) \;=\; \sum_{k=0}^{\left[\frac{n}{q}\right]} A^*_{\nu+qk,\; n-qk}\, C^{\mu,\nu}_k\, \Omega_{\mu+pk}(y_1, \cdots, y_s) z^k. \tag{1.9}$$

Special case: For $s = 1$, the bilateral generating relations (1.8) can be written as:

$$\sum_{n=0}^{\infty} S^*_{\nu+n}(x) Y^{p,\mu}_{n,q,\nu}(y; z) t^n = f^*(x,t)\{g(x,t)\}^{-\nu}\; G^{p,\mu}_{q,\nu}\Big[h(x,t); y; z\Big\{\frac{t}{g(x,t)}\Big\}^q\Big] \tag{1.10}$$

where

$$Y^{p,\mu}_{n,q,\nu}(y; z) \;=\; \sum_{k=0}^{\left[\frac{n}{q}\right]} A^*_{\nu+qk,\; n-qk}\, C^{\mu,\nu}_k\, \Theta_{\mu+pk}(y) z^k \tag{1.11}$$

and

$$G^{p,\mu}_{q,\nu}[x; y; t] \;=\; \sum_{n=0}^{\infty} C^{\mu,\nu}_n\, S^*_{\nu+qn}(x)\, \Theta_{\mu+pn}(y) t^n \tag{1.12}$$

$C^{\mu,\nu}_n \neq 0$, $\Theta_\mu(y) \neq 0$ is an arbitrary function of y, p and q are positive integers, μ and ν are arbitrary complex numbers.

Konhauser polynomials of second kind (Srivastava [7]):

It is denoted by the symbol $Z^\alpha_n(x; k)$ and defined as,

$$Z^\alpha_n(x; k) \;=\; \frac{\Gamma(kn+\alpha+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^{kj}}{\Gamma(kj+\alpha+1)} \tag{1.13}$$

where k is a positive integer.

$L_n^{\alpha,\beta}(x)$ *Polynomials* (Prabhakar and Suman [2]): It is defined as:

$$L_n^{\alpha,\beta}(x) = \frac{\Gamma(\alpha n+\beta+1)}{n!}\sum_{k=0}^{n}\frac{(-n)_k\, x^k}{\Gamma(\alpha k+\beta+1)\, k!} \tag{1.14}$$

$\text{Re}(\beta) > -1$ and α is any complex number with $\text{Re}(\alpha) > 0$.

The following relations are also established by Prabhakar and Suman [2]

$$L_n^{1,\beta}(x) = Z_n^{\beta}(x;1) = L_n^{\beta}(x). \tag{1.15}$$

2 Main Result

On comparing (1.5) with (1.6) and replacing ν by m (a non-negative integer), we get

$$A^*_{m,n} = \begin{pmatrix} n+m \\ n \end{pmatrix}, \quad S^*_{m+n}(x) = V_{n+m}^{(\alpha,\beta)}(x;a,k,s), \quad g(x,t) = 1,$$

$$S^*_m(h(x,t)) = V_n^{(\alpha,\beta)}\left(x(1-ax^at)^{-1/a};a,k,s\right),$$

$$f^*(x,t) = (1-ax^at)^{-\left(\frac{\beta+s}{a}\right)}\,\frac{E_\alpha\{p_k(x)\}}{E_\alpha[p_k\{x(1-ax^at)^{-1/a}\}]}$$

and applying Theorem 2 (Srivastava [6]) yields bilateral generating relation for the sequence of function $V_n^{(\alpha,\beta)}(x;a,k,s)$ which is given as: For $s \geq 1$

$$\sum_{n=0}^{\infty} V_{n+m}^{(\alpha,\beta)}(x;a,k,s)W_{n,q,m}^{p,\mu}(y_1,\cdots,y_s;z)t^n = (1-ax^at)^{-\left(\frac{\beta+s}{a}\right)}$$

$$\times\,\frac{E_\alpha\{p_k(x)\}}{E_\alpha[p_k\{x(1-ax^at)^{-1/a}\}]}\,F_{q,m}^{p,\mu}\{x(1-ax^at)^{-1/a};y_1,\cdots,y_s;zt^q\} \tag{2.1}$$

where

$$W_{n,q,\nu}^{p,\mu}(y_1,\cdots,y_s;z) = \sum_{j=0}^{\left[\frac{n}{q}\right]} A^*_{m+qj,\; n-qj}\, A_j\, \Omega_{m+pj}(y_1,\cdots,y_s)z^j$$

$$= \sum_{j=0}^{\left[\frac{n}{q}\right]} \begin{pmatrix} n+m \\ n-qj \end{pmatrix} A_j\, \Omega_{m+pj}(y_1,\cdots,y_s)z^j \tag{2.2}$$

and

$$F^{p,\mu}_{q,m}[x; y_1, \cdots, y_s; t] = \sum_{n=0}^{\infty} A_n \, V^{(\alpha,\beta)}_{m+qn}(x; a, k, s) \, \Omega_{\mu+pn}(y_1, \cdots, y_s) t^n \tag{2.3}$$

where $A_n \neq 0$, μ is an arbitrary complex number, p and q are positive integers.

3 Applications

(i) If $s = 1$, then bilateral generating relation (2.1) reduces to,

$$\sum_{n=0}^{\infty} V^{(\alpha,\beta)}_{n+m}(x; a, k, s) W^{p,\mu}_{n,q,m}(y; z) t^n = (1-ax^a t)^{-\left(\frac{\beta+s}{a}\right)}$$

$$\times \frac{E_\alpha\{p_k(x)\}}{E_\alpha[p_k\{x(1-ax^a t)^{-1/a}\}]} \, G^{p,\mu}_{q,m}\{x(1-ax^a t)^{-1/a}; y; zt^q\} \tag{3.1}$$

where

$$W^{p,\mu}_{n,q,\nu}(y; z) = \sum_{j=0}^{\left[\frac{n}{q}\right]} A^*_{m+qj,\ n-qj} \, A_j \, \Theta_{m+pj}(y) z^j$$

$$= \sum_{j=0}^{\left[\frac{n}{q}\right]} \binom{n+m}{n-qj} A_j \, \Theta_{m+pj}(y) z^j \tag{3.2}$$

and

$$G^{p,\mu}_{q,m}[x; y; t] = \sum_{n=0}^{\infty} A_n \, V^{(\alpha,\beta)}_{m+qn}(x; a, k, s) \, \Theta_{\mu+pn}(y) t^n \tag{3.3}$$

where $A_n \neq 0$, $\Theta_\mu(y) \neq 0$ is an arbitrary functions of y, μ is an arbitrary complex number, p and q are positive integers.

(ii) Choosing $m = 0$, $q = 1$, $\Omega_\mu(y_1, \cdots, y_s) \equiv 1$, $A_n = \dfrac{(-1)^n}{\Gamma(\gamma n + \delta + 1)}$ in (2.1) and using (1.14), we get following new bilateral generating relation for the sequence of function $V^{(\alpha,\beta)}_n(x; a, k, s)$ as

$$\sum_{n=0}^{\infty} V_n^{(\alpha,\beta)}(x;a,k,s)W_{n,1,0}^{p,\mu}(y_1,\cdots,y_s;z)t^n \;=\; (1-ax^a t)^{-\left(\frac{\beta+s}{a}\right)}$$

$$\times\; \frac{E_\alpha\{p_k(x)\}}{E_\alpha[p_k\{x(1-ax^a t)^{-1/a}\}]}\, F_{1,0}^{p,\mu}\{x(1-ax^a t)^{-1/a};y_1,\cdots,y_s;zt\} \tag{3.4}$$

where

$$W_{n,1,0}^{p,\mu}(y_1,\cdots,y_s;z) \;=\; \sum_{j=0}^{n} A^*_{j,\;n-j}\, A_j\, z^j \;=\; \sum_{j=0}^{n} \binom{n}{n-j} A_j\, z^j$$

$$=\; \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{\Gamma(\gamma n+\delta+1)}\, z^j \;=\; \frac{n!}{\Gamma(\gamma n+\delta+1)}\, L_n^{\gamma,\delta}(z) \tag{3.5}$$

and

$$F_{1,0}^{p,\mu}[x;y_1,\cdots,y_s;zt] = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\gamma n+\delta+1)}\, V_n^{(\alpha,\beta)}(x;a,k,s)\, z^n t^n \tag{3.6}$$

hence (3.4) becomes

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(\gamma n+\delta+1)}\, V_n^{(\alpha,\beta)}(x;a,k,s)\, L_n^{\gamma,\delta}(z)t^n \;=\; (1-ax^a t)^{-\left(\frac{\beta+s}{a}\right)}$$

$$\times\; \frac{E_\alpha\{p_k(x)\}}{E_\alpha[p_k\{x(1-ax^a t)^{-1/a}\}]}\frac{(-1)^n}{\Gamma(\gamma n+\delta+1)}\, V_n^{(\alpha,\beta)}\{x(1-ax^a t)^{-1/a};a,k,s)\}z^n t^n. \tag{3.7}$$

Therefore, we arrived at the conclusion that (2.1), (3.1) and (3.7) are very interesting bilateral generating relations and supposed to be new to the literature.

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