On the Trinomial Arcs $J(p,k,r,n)$

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Abstract
We study the trinomial arcs $J(p,k,r,n)$ and we prove the monotonicity of this category of arcs.

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1 Introduction and Preliminaries

In [2], it was established that the trinomial arcs $I(p,k,r,n)$ are monotonic, a question which was pointed out in [1]. According to [2], $I(p,k,r,n)$ is the set of roots of the trinomial equation

$$z^n = \alpha z^k + (1 - \alpha) \quad (1)$$

where $\alpha$ is a real number between 0 and 1, $z = \rho e^{i\theta}$ is a complex variable, $n$ and $k$ are two integers such that $k = 1, 2, ..., n-1$. Let us recall that an angle $\theta$ is called feasible for equation (1) with $0 < \alpha < 1$ if $\text{sign} (\sin n\theta) = \text{sign} (\sin k\theta) = -\text{sign} (\sin (n-k)\theta)$. The feasible angles $\theta$ for the arcs $I(p,k,r,n)$ belong to the interval $[\arg (\gamma), \arg (\delta)]$ such that $\gamma$ is an $n^{th}$ root of unity and $\delta$ is both a $k^{th}$ root of $-1$ and an $(n-k)^{th}$ root of unity. In view of [2], there exists another type of trinomial arcs inside the unit disk, which we will denote by $J(p,k,r,n)$, such that the feasible angles belong to $[\arg (\delta), \arg (\gamma')]$ where $\gamma'$ is an $n^{th}$ root of unity and $\delta$ is equidistant from $\gamma$ and from $\gamma'$. If we set $\arg (\gamma) = 2\pi r/n$ where $r$ is a nonzero integer, so $\arg (\gamma') = 2(r+1)\pi/n$. Moreover, we can put $\arg (\delta) = (2p+1)\pi/k$ where $p$ is an integer.

The study of the behavior of the trinomial arcs $J(p,k,r,n)$ is a problem pointed out in [2] and before that in [1]. In the present work, this problem will be completely solved.
Note that the continuous arcs $J(p,k,r,n)$ illustrated in the figure below can be expressed in polar coordinates $(\rho, \theta)$ by a function $\rho(\theta)$. The main purpose of this paper is to show that $\rho(\theta)$ is an increasing function. Applying the symmetry map $z \rightarrow \overline{z}$, the upper and lower half-planes are symmetrical. Thus, we restrict our study to the upper half-plane.

In [2], Lemma 3.1 and Remark 3.2 allow us to affirm that for any trinomial arc $J(p,k,r,n)$, the integer $k$ verify that $k = (2p + 1)n/(2r + 1)$ where the integers $p$ and $r$ satisfy the condition $r \geq p + 1$ and such that $\text{arg} (\delta) = (2p + 1)\pi/k$ and $\text{arg} (\gamma') = 2(r + 1)\pi/n$. The trajectories of roots of (1) with $0 < \alpha < 1$ are linear when $n = 2$. Hence, we define $J(p,k,r,n)$ as follows.

Proposition 1.1 Let $n$ be an integer larger than or equal to 3 and $\alpha$ be a real number between 0 and 1. In equation (1) with $k = (2p + 1)n/(2r + 1)$ is an integer and $p$ and $r$ are two integers such that $r > p$, any angle of the interval $[(2p + 1)\pi/k, 2(r + 1)\pi/n]$ is feasible.

Proof. Assume that $k = (2p + 1)n/(2r + 1)$ is an integer. Let us consider an angle $\theta$ such that $(2p + 1)\pi/k < \theta < 2(r + 1)\pi/n$. Because $(2p + 1)\pi/k = (2r + 1)\pi/n$, we find that $(2r + 1)\pi < n\theta < (2r + 2)\pi$ and that $\sin n\theta < 0$. Also, one can see immediately that $(2p + 1)\pi < k\theta < 2(r + 1)(2p + 1)\pi/(2r + 1)$. Since $r \geq p + 1$, it yields that $2(r + 1)(2p + 1)\pi/(2r + 1) < (2p + 2)\pi$ and that $\sin k\theta < 0$. At last, one has $4(r - p)\pi/(2r + 1)< 4(r + 1)(r - p)\pi/(2r + 1)$. As $4(r + 1)(r - p)\pi/(2r + 1) < [2(r - p) + 1]\pi$, we conclude that $\sin(n - k)\theta > 0$. Therefore, the angle $\theta$ is feasible and the proof is achieved.
2 Main Results

We shall make use of the following proposition.

**Proposition 2.1** \(\rho(\theta)\) is differentiable for all the trinomial arcs \(J(p, k, r, n)\).

**Proof.** Let be \(J(p, k, r, n)\) a trinomial arc. First, divide equation (1) by \(z^n\). When \(\theta \neq s\pi/(n-k), s \in \mathbb{N}\), one can have \(\rho^k(\theta) = (1-1/\alpha) \sin n\theta / \sin(n-k)\theta\). By Proposition 1.1, any feasible angle \(\theta\) is such that \(\sin n\theta < 0\) and \(\sin(n-k)\theta > 0\). Because \(0 < \alpha < 1\), the function \(\theta \mapsto (1-1/\alpha) \sin n\theta / \sin(n-k)\theta\) is well-defined. Also, this function is differentiable and positive. So, \(\rho(\theta) = [(1-1/\alpha) \sin n\theta / \sin(n-k)\theta]^{1/k}\) is differentiable. Hence, \(d\rho/d\theta\) exists and it’s well-defined. Thus, we achieve the proof.

Now, for \(z = \rho e^{i\theta}\) in equation (1), one can find that \(\rho^e^{i\theta} = \alpha \rho^k e^{i(k\theta)} + 1 - \alpha\). Separating real and imaginary parts, one has

\[
\rho^n - k \sin n\theta - \rho^n \sin (n-k)\theta = \sin k\theta. \tag{2}
\]

Let us differentiate both sides of this equation with respect to \(\theta\). Then

\[
\left[ (n-k) \rho^{n-k-1} \sin n\theta - n \rho^{n-1} \sin(n-k)\theta \right] d\rho/d\theta
= k \cos k\theta + (n-k) \rho^n cos(n-k)\theta - n \rho^{n-k} \cos n\theta.
\]

If we suppose that \(d\rho/d\theta = 0\), we obtain that

\[
\begin{cases}
  k \cos k\theta + (n-k) \rho^n \cos(n-k)\theta - n \rho^{n-k} \cos n\theta = 0 \\
  \rho^n - k \sin n\theta - \rho^n \sin (n-k)\theta - \sin k\theta = 0
\end{cases}
\]

This system is equivalent to the following

\[
\begin{cases}
  Z(\theta) \cdot \rho^{n-k} = X(\theta) \\
  Z(\theta) \cdot \rho^n = Y(\theta)
\end{cases} \tag{3}
\]

with

\[
Z(\theta) = n \sin k\theta - k \sin n\theta \cos(n-k)\theta \\
X(\theta) = n \sin k\theta \cos(n-k)\theta - k \sin n\theta \\
Y(\theta) = (n-k) \sin k\theta \cos n\theta - k \sin(n-k)\theta.
\]

As consequence of (3), one can deduce that

\[
Z(\theta) [\rho^n - \rho^{n-k}] = S(\theta) [1 - \cos k\theta] \tag{4}
\]

with

\[
S(\theta) = -[n \sin(n-k)\theta + (n-k) \sin n\theta].
\]

In the rest of this note, we have to contradict the hypothesis \(d\rho/d\theta = 0\) for the family of arcs \(J(p, k, r, n)\). Thus, we shall make use of the following propositions.
Proposition 2.2 Assume that $J(p, k, r, n)$ is a trinomial arc. We have $Z(\theta) < 0$ for all the feasible angles.

Proof. Let $J(p, k, r, n)$ be a trinomial arc and $\theta$ be a feasible angle. Because $\sin n\theta < 0$ by Proposition 1.1 and $\cos(n - k)\theta \leq 1$, one can find immediately that $Z(\theta) \leq T(\theta)$, with $T(\theta) = n \sin k\theta - k \sin n\theta$. The values of $\theta$ for which $T'(\theta) = 0$ are of the form $\theta = 2j\pi/(n - k)$ or of the form $\theta = 2j\pi/(n + k)$, where $j \in \mathbb{N}$. However, $2j\pi/(n - k) \in [(2p + 1)\pi/k, 2(r + 1)\pi/n]$ if and only if $r - p < j < 2(r + 1)(r - p)/(2r + 1)$. But, the inequality $2(r + 1)(r - p)/(2r + 1) < r - p + 1/2$ contradicts the fact that $j$ is an integer. Moreover, $2j\pi/(n + k) \in [(2p + 1)\pi/k, 2(r + 1)\pi/n]$ is equivalent to $r + p + 1 < j < 2(r + 1)(r + p + 1)/(2r + 1)$. Since $r > p$, one gets $2(r + 1)(r + p + 1)/(2r + 1) < r + p + 2$, which isn’t possible. This allows us to conclude that $T(\theta)$ is monotonic on the interval $[(2p + 1)\pi/k, 2(r + 1)\pi/n]$. Lastly, observing that $T((2p + 1)\pi/k) = 0$ and that $T(2(r + 1)\pi/n) < 0$, we deduce that $T(\theta) < 0$ and that $Z(\theta) < 0$ for any feasible angle $\theta$.

Proposition 2.3 Suppose that $J(p, k, r, n)$ is a trinomial arc. We have $S(\theta) < 0$ for all the feasible angles.

Proof. Assume that $J(p, k, r, n)$ is a trinomial arc and that $\theta$ is a feasible angle. Estimating $S(\theta)$ at the bounds of the interval of feasible angles, one can obtain that $S((2p + 1)\pi/k) = 0$ and that $S(2(r + 1)\pi/n) < 0$. In order to show that $S(\theta) < 0$ on $[(2p + 1)\pi/k, 2(r + 1)\pi/n]$, we have to prove that $S(\theta)$ is monotonic on this interval. The roots of the equation $S'(\theta) = 0$ are of the form $\theta = (2j - 1)\pi/k$ or of the form $\theta = (2j + 1)\pi/(2n - k)$ where $j \in \mathbb{N}$. As for the first solution, $(2j - 1)\pi/k$ is feasible if and only if $p + 1 < j < [(r + 1)(2p + 1)/(2r + 1)] + 1/2$. Because $r > p$, it yields that $[(r + 1)(2p + 1)/(2r + 1)] + 1/2 < p + 3/2$. This can not occur as $j$ is an integer. For the second solution, $(2j + 1)\pi/(2n - k)$ is feasible if and only if $2r - p < j < [(r + 1)(4r - 2p + 1)/(2r + 1)] - 1/2$. But the inequality $[(r + 1)(4r - 2p + 1)/(2r + 1)] - 1/2 < 2r - p + 1$ contradicts the fact that $j$ is an integer. This clearly means that $S(\theta)$ is a monotonic function. Thus, we achieve the proof.

Theorem 2.4 Let $J(p, k, r, n)$ be a trinomial arc. For all the feasible angles $\theta$, the function $\rho(\theta)$ is increasing.

Proof. Let $\theta$ be a feasible angle on the interval $[(2p + 1)\pi/k, 2(r + 1)\pi/n]$. By first, the question is to prove that $\rho(\theta)$ is a monotonic function. Applying the propositions 2.2 and 2.3, we have $Z(\theta) < 0$ and $S(\theta) < 0$. According to equation (4), we find that $Z(\theta)\rho^n = S(\theta)[1 - \cos k\theta]$. Hence, one can deduce that $\rho^n > \rho^{n-k}$. This last inequality provides a contradiction with the
fact that $\rho < 1$. Thus, the hypothesis $d\rho/d\theta = 0$ is not possible for the arcs $J(p, k, r, n)$. Otherwise, to estimate $\rho(\theta)$ at the angle $2(r+1)\pi/n$, let us replace $\theta$ by $2(r+1)\pi/n$ in equation (2). It follows that $(\rho^n - 1) \sin [2(r+1)\pi k/n] = 0$. Because $(2p+1)\pi < 2(r+1)\pi k/n < 2(p+1)\pi$, one has $\sin [2(r+1)\pi k/n] \neq 0$. This implies that $\rho[2(r+1)\pi/n] = 1$. Remarking that $\rho(\theta)$ don’t exceed 1 as the arcs $J(p, k, r, n)$ are inside the unit disk, one can have in conclusion of the monotonicity of $J(p, k, r, n)$ that $\rho(\theta)$ is an increasing function.

3 Conclusion

The family of arcs $J(p, k, r, n)$ is one of several families of trinomial arcs solutions of the equation (1). The union of these arcs $J(p, k, r, n)$ studied in this note is a fractal set. Though not as complicated as the Mandelbrot set [3], it’s sufficiently irregular to be considered as an object in computer graphics. In the present paper, it’s established that the function $\rho(\theta)$ is monotonic and more precisely is increasing for all trinomial arcs $J(p, k, r, n)$. Using this result, we hope that it will be possible in the future to estimate the fractal dimension of this fractal set.

References


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