

# Automatic Continuity of Separating Linear Isomorphisms on a Class of Topological Algebras

E. Ansari-Piri <sup>1</sup> and M. Aghajani

Department of Mathematics, Faculty of Science  
University of Guilan, P.O.Box 1914, Rasht, Iran  
e.ansari@guilan.ac.ir  
aghajani\_238@yahoo.com

## Abstract

The continuity and general form of a separating linear isomorphisms  $A : C(T) \rightarrow C(S)$ , when  $T$  and  $S$  are compact topological spaces, have been studied in last few years. Recently, we have changed the conditions on  $T$  and  $S$  and proved that the separating linear isomorphism  $A : C_c(T) \rightarrow C_c(S)$  has a general form. In this note, we prove the continuity of such isomorphisms.

**Mathematics Subject Classification:** Primary 46H

**Keywords:** Separating linear isomorphism, continuous linear isomorphism

## 1 Introduction

When  $T$  and  $S$  are compact topological spaces, in [3] it is proved that every separating linear isomorphism  $A : C(T) \rightarrow C(S)$  is continuous and has a general form. In [4], the continuity and general form of bi-separating linear maps between the algebras  $B(E)$  of all linear continuous maps on a Banach space  $E$  is also discussed, where by a bi-separating linear isomorphism we mean a separating linear isomorphism with a separating linear inverse. The continuity and general form of bi-separating linear isomorphisms between standard subalgebras of bounded operators on Frechet spaces is considered in [1]. In [2] The authors also have recently proved that, where  $T$  is non-compact locally compact Hausdorff and  $S$  is only Hausdorff space, then every separating linear isomorphism  $A : C_c(T) \rightarrow C_c(S)$  has a general form. In this note we prove the continuity of such isomorphisms, if moreover  $S$  is a locally compact and completely regular space.

---

<sup>1</sup>Corresponding author

Let us recall that a linear map  $A$  between the algebras  $X$  and  $Y$  is said to be separating if  $xy = 0 \Rightarrow A(x)A(y) = 0$  for all  $x, y \in X$ .

## 2 The continuity of separating linear isomorphisms

Let  $T$  be a non-compact, normal, locally compact and  $S$  be a Hausdorff space. Suppose  $A : C_c(T) \rightarrow C_c(S)$  is a linear separating map. In [2] it is proved that there are three subsets  $S_1, S_2, S_3$  for  $S$  such that  $S = S_1 \cup S_2 \cup S_3$  where  $S_3$  is closed and there is a continuous map  $\varphi : S_1 \cup S_2 \rightarrow T$  and a non vanishing continuous map  $\chi : S_1 \rightarrow \mathbb{C}$  such that for every  $f \in C_c(T)$ , and every  $s \in S_1$  :

$$Af(s) = \chi(s) \cdot f \circ \varphi(s),$$

and  $Af \equiv 0$  on  $S_3$ . Moreover, for all  $f \in C_c(T)$ , if  $(\text{supp}f) \cap \varphi(S_2) = \emptyset$ , then  $Af|_{S_2} \equiv 0$ .

Here we prove that if moreover  $S$  is a locally compact and completely regular space, then the separating linear isomorphism  $A$  is continuous.

**Proposition 2.1** *The two sets  $S_1$  and  $S_2$  have the following properties:*

1. *The subset  $S_1$  of  $S$  is closed,*
2.  *$\varphi(S_2) \cap \text{int}K$  is finite for every compact subset  $K$  of  $T$ .*

**Proof.** *Following the proof of [2; theorem 3.1], for  $s \in S$ , if we denote by  $\delta_s$  the evaluation functional of  $C_c(S)$  at the point  $s$  then, for  $f \in C_c(T)$ ,  $\delta_s \circ A(f) = Af(s)$ , and  $S_3 = \{s \in S : \delta_s \circ A \equiv 0\}$ ,  $S_2 = \{s \in S : \delta_s \circ A \text{ is discontinuous}\}$  and  $S_1 = S \setminus (S_2 \cup S_3)$ , and so  $S_1 = \{s \in S : \delta_s \circ A \text{ is non-zero and continuous}\}$ .*

*To prove (1), let  $(s_\alpha)_{\alpha \in I}$  be a net in  $S_1$  which converges to  $s \in S$ . Since  $T$  is locally compact and  $\varphi(s) \in T$ , so  $\exists f \in C_c(T)$ ;  $f(\varphi(s)) \neq 0$ . By the continuity of the map  $f$ , we have  $f(\varphi(s_\alpha)) \rightarrow f(\varphi(s))$ . Therefore, there exists  $\alpha_0 \in I$  such that for all  $\alpha$  with  $\alpha \geq \alpha_0$  we have  $f(\varphi(s_\alpha)) \neq 0$ . Now,  $\chi(s_\alpha) \neq 0$  so  $Af(s_\alpha) \neq 0$  for all  $\alpha \geq \alpha_0$ . By regarding to  $C_c(T) = \ker \delta_{s_\alpha} \circ A \oplus \langle f \rangle$ , for all  $\alpha \geq \alpha_0$  we can assume the existence of an scalar  $\beta_\alpha$  and a map  $h_\alpha \in \ker \delta_{s_\alpha} \circ A$  for every  $g \in C_c(T)$  such that  $g = h_\alpha + \beta_\alpha f$ , thus  $g(\varphi(s_\alpha)) = h_\alpha(\varphi(s_\alpha)) + \beta_\alpha f(\varphi(s_\alpha))$ . Since  $s_\alpha \in S_1$ , so  $\ker \delta_{s_\alpha} \circ A = K_{s_\alpha}$  where  $K_s = \{f \in C_c(T); f(\varphi(s)) = 0\}$  for every  $s \in S$ . We put  $\beta_\alpha = \frac{g(\varphi(s_\alpha))}{f(\varphi(s_\alpha))}$ , and therefore we have  $g(\varphi(s_\alpha)) = h_\alpha(\varphi(s_\alpha)) + \frac{g(\varphi(s_\alpha))}{f(\varphi(s_\alpha))} \cdot f(\varphi(s_\alpha))$  and also  $Ag(s_\alpha) = Ah(s_\alpha) + \frac{g(\varphi(s_\alpha))}{f(\varphi(s_\alpha))} \cdot Af(s_\alpha)$ . Now we put  $\beta = \frac{Af(s)}{f(\varphi(s))}$ , then we have  $Ag(s) = \beta \cdot g(\varphi(s))$ . In other words, we get  $\delta_s \circ A(g) = \beta \cdot (g \circ \varphi)(s)$  for every  $g \in C_c(T)$ , and so  $\delta_s \circ A$  is continuous, which means  $s \in S_1$ .*

*For (2), we suppose that there exists a compact subset  $K$  of  $T$  such that  $\varphi(S_1) \cap \text{int}K$  is infinite. Since  $K$  is compact so there exists a sequence  $(\varphi(s_n))_{n \in \mathbb{N}}$*

of distinct elements of  $\text{int}K$  with  $s_n \in S_2$  for  $n \in N$  and  $(\varphi(s_n))_{n \in N}$  converges to a point of  $K$ . Therefore we can assume that  $(U_n)_{n \in N}$  is a pair wise disjoint sequence of open subsets of  $K$  such that  $\varphi(s_n) \in U_n \subseteq K$  for every  $n \in N$ . We can also assume that the closure of every  $U_n$  is compact. Let  $V_n$  be a neighborhood of  $\varphi(s_n)$  with  $V_n \subseteq U_n$  and  $\overline{V_n}$  is compact. Thus there exists a map  $g_n \in C_c(T)$  such that  $0 \leq g_n \leq 1$ ,  $g_n|_{V_n} \equiv 1$  and  $\text{coz}(g_n) \subseteq U_n$  for each  $n \in N$ . On the other hand, since  $\delta_{s_n} \circ A$  is discontinuous, there exists a map  $h_n \in C_c(T)$  with  $\sup\{|h_n(t)|; t \in K\} \leq 1$  and such that  $|\delta_{s_n} \circ A(h_n)| = |Ah_n(s_n)| > n^3$  for all  $n \in N$ . We put  $f_n = 1/n^2 \cdot h_n \cdot g_n$ . Since  $g_n \equiv 1$  on  $V_n$  then, we have that  $|Af_n(s_n)| = 1/n^2 |Ah_n(s_n)| > n$ . So  $|Af_n(s_n)| > n$  for each  $n \in N$ . By regarding to  $|f_n| \leq 1/n^2$  we define  $f = \sum_{n \in N} f_n$ . Since  $\text{supp}f \subseteq \cup \text{supp}f_n \subseteq K$ , so  $f \in C_c(T)$ . On the other hand, as  $(U_n)$  is pair wise disjoint and  $\text{coz}(f_n) \subseteq U_n$ , for all  $n \in N$ , then  $Af_n|_{\varphi^{-1}(U_m)} \equiv 0$  for  $n \neq m$ . Thus  $|Af(s_n)| = |Af_n(s_n)| > n$  for all  $n \in N$ , which is a contradiction with  $Af$  is bounded.

**Theorem 2.2** Let  $A : C_c(T) \rightarrow C_c(S)$  be a separating linear isomorphism . Then  $A$  is continuous and  $Af(s) = \chi(s) \cdot f(\varphi(s))$  for all  $s \in S$  and  $f \in C_c(T)$ .

**Proof.** Since  $A$  is injective, by [2; corollary 3.3]  $\overline{\varphi(S)} = T$ . Let  $t \in T$  be a limit point and  $U$  be a neighborhood of  $t$  with compact closure. By proposition (2.1),  $U \cap \varphi(S_2)$  is finite where  $U \cap \varphi(S)$  is infinite and therefore  $U \cap \varphi(S_1)$  is infinite . Since  $\varphi(S_2)$  consists only of the limit points of  $T$ , this implies that  $\overline{\varphi(S_1)} = T$ .

For every  $f \in C_c(T)$  and  $s \in S_1$  we have  $Af(s) = \chi(s) \cdot f(\varphi(s))$  . Now, if  $Af(s) = 0$  for  $s \in S_1$  and some  $f \in C_c(T)$ , since  $\chi(s) \neq 0$  so  $f(\varphi(s)) = 0$ . Therefore, if  $Af|_{S_1} = 0$ , then  $f(\varphi(S_1)) \equiv 0$  i.e.  $f|_{(\varphi(S_1))} \equiv 0$  and since  $\overline{\varphi(S_1)} = T$ , so  $f = 0$  on  $T$ . Now, we claim  $S_2 = \emptyset$ . For if  $s \in S_2$ , then  $\{s\} \cap S_2 = \emptyset$  and since  $S_1$  is closed by Urysohn's lemma there exists  $g \in C_c(S)$  with  $g(s_0) = 1$  and  $g = 0$  on  $S_1$ . Since  $A$  is bijection and  $g$  is a non-zero, so there exists a non-zero  $f$  in  $C_c(T)$  with  $Af = g$ , i.e.  $Af(s) = 1$  and  $Af|_{(S_1)} \equiv 0$  which is a contradiction.

We close the paper with a remark which says that if we consider  $A : C_0(T) \rightarrow C_0(S)$  with the same conditions then  $A$  has a general form, but we believe the continuity of  $A$  is still open.

**Remark.** Let  $A : C_0(T) \rightarrow C_0(S)$  be a linear separating map. Suppose  $A_1 = A|_{C_0(T)}$  and  $S_0 = \{s \in S : \delta_s \circ A \text{ is continues on } C_0(T)\}$ . Since the proof of Theorem (3.1) of [2] valid if we replace  $C_c(S)$  by  $C_0(S)$ , so there exists a continuous map  $\varphi : S_1 \cup S_2 \rightarrow T$  and a non vanishing continuous map  $\chi : S_1 \rightarrow C$  such that  $A_1f(s) = \chi(s) \cdot f(\varphi(s))$  for every  $f \in C_c(T), s \in S_0$ . Now , let  $f \in C_0(T)$ , and choose  $(f_n)$  from  $C_c(T)$  with  $f_n \rightarrow f$  in  $C_0(T)$  and let

$s \in S_0$ . Then  $\delta_s \circ A(f_n) \rightarrow \delta_s \circ A(f)$  and  $\delta_s \circ A(f) = \lim_{n \rightarrow \infty} \chi(s) \cdot f_n(\varphi(s)) = \chi(s) \cdot f(\varphi(s))$ .

## References

1. E.Ansari-Piri and M.Aghajani, Bi-separating linear isomorphisms on a class of topological algebras, Far East Journal of Mathematical sciences(FMJS), Vol.28,No.2,2008,483-487.
2. E.Ansari-Piri and M.Aghajani, The Separating linear operators in Topological Algebras, Korean annals of Mathematics (to appear).
3. K.Jarosz, Automatic continuity of separating linear isomorphisms, Canad. Math.Bull.vol.33(2),1990.
4. J.Araujo and K.Jarosz, Bi-separating maps between operators, J. Math. Anal. Appl. 282(2003) 48-55.

**Received: September 21, 2008**