

Dynamics of Generalized Time Dependent Predator Prey Model with Nonlinear Harvesting

Teffera M. Asfaw

University of South Florida
Department of Mathematics
Tampa, FL 33620-5700, USA
tefferam@yahoo.com

Abstract

In this paper, we consider the system of nonlinear ordinary differential equation

$$\begin{aligned}\dot{x} &= x(t)(1 - x(t)) - a(t)f(x(t), y(t)) - h(t)\gamma(x(t)) \\ \dot{y} &= y(t)(-d(t) + b(t)g(x(t), y(t))) - k(t)\rho(y(t))\end{aligned}$$

with $t \geq 0$ and initial conditions $x(0) = x_0 > 0$, $y(0) = y_0 > 0$ and a, b, c, d, h, k and f, g are continuous from $[0, \infty)$ to $(0, \infty)$, ρ, γ are continuous from $[0, \infty)$ to $[0, \infty)$. Boundedness of solution (often called permanence) of this system is proved under suitable assumptions on the functions involved. Different examples of predator prey models are discussed as an application of the result. At last equilibria of some models are computed and their stability is analyzed.

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1. Introduction

The above system is a generalization of the population model often called Lotka Votera population model. D. Xiao and L. Jennings ([6]) studied the dynamics of constant harvesting of the prey using the system

$$\begin{aligned}\dot{x} &= x(1 - x) - \frac{axy}{y + x} - h \\ \dot{y} &= y(-d + \frac{bx}{y + x})\end{aligned}$$

where a, b, d and h are all positive constants. B. Leard, C.Lewis and J.Rebeza([3]) studied the ratio dependent predator prey model with different non-constant harvesting of the prey which can take the following form

$$\begin{aligned}\dot{x} &= x(1-x) - \frac{axy}{y+x} - H(x) \\ \dot{y} &= y\left(-d + \frac{bx}{y+x}\right)\end{aligned}$$

where $H(x) = hx$ and $H(x) = \frac{hx}{c+x}$ where h and c are positive constants. These population models are known as the Michaelis-Menten ratio dependent predator prey models. In ([3]), the existence of equilibrium points and their stability properties are studied. Conditions of extinction and coexistence of the two species are discussed. The details can be found in ([3]).

In section 2, boundedness of solution of the system is proved. Moreover, the result will be applied in analyzing different predator prey models. Sufficient conditions of mutual coexistence and extinction of the species is given as an application. In particular, it is proved that the survival of both of the species is mainly controlled by the properties of the functions a, b, d, h and k rather than the functions f, g, γ and ρ . It is shown that the species will never extinct (mutual co-existence is possible) even though both populations are harvested aggressively.

In section 3, two ratio dependent Michaelis-Menten ratio dependent predator prey models with harvesting functions of both the prey and predator species are considered. Equilibria of some systems are computed and their stability properties are analyzed.

2. Boundedness of solution of the system

We consider the system of nonlinear ordinary differential equation

$$\begin{aligned}\dot{x} &= x(1-x) - a(t)f(x, y) - h(t)\gamma(x) \\ \dot{y} &= y(-d(t) + b(t)g(x, y)) - k(t)\rho(y)\end{aligned}\quad (*)$$

with initial conditions $x(0) = x_0 > 0$, $y(0) = y_0 > 0$ and a, b, c, d, h and k are continuous functions from $[0, \infty)$ to $(0, \infty)$, ρ and γ are continuous functions from $[0, \infty)$ to $[0, \infty)$ and f and g are continuous functions from $[0, \infty)$ to $(0, \infty)$.

Definition 1 . *The system (*) is called permanent if there exists a compact region $\Omega \subseteq R_+^2$ such every solution of the system will eventually enter and remain in Ω . In other words, there are positive constants α, β, γ and λ such that every solution (x, y) starting at (x_0, y_0) lies in $[\alpha, \beta] \times [\gamma, \lambda]$ for large t .*

Suppose the following are satisfied.

$H_1)$ $f, g : (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$ are continuous with g bounded, i.e. there exist $\mu > 0$ such that $g(x, y) \leq \mu$ for all (x, y) .

$H_2)$ $\gamma, \rho : (0, \infty) \rightarrow [0, \infty)$ are increasing such that $\gamma(x) \geq x^2$ for all x .

$H_3)$ $a, b, h, d, k : [0, \infty) \rightarrow (0, \infty)$ are continuous such that

$$\int_0^\infty \frac{1}{h(s)} ds < \infty \quad \text{and} \quad \int_0^\infty b(s) ds < \infty.$$

Thus we prove the following Theorems.

Theorem 1 . Suppose (x, y) is a solution of the system $(*)$. Assume H_1, H_2 and H_3 are satisfied. Then

$$\limsup_{t \rightarrow +\infty} x(t) < +\infty \quad \text{and} \quad \limsup_{t \rightarrow +\infty} y(t) < +\infty.$$

Proof: Considering $(*)$, it is easy to see that

$$\begin{aligned} \dot{x} &= x(1-x) - a(t)f(x, y) - h(t)\gamma(x) \\ &\leq x(1-x) - a(t)f(x, y) - h(t)x^2 \\ &= x - (1+h(t))x^2 - a(t)f(x, y) \\ &\leq \frac{1}{4(1+h(t))} - a(t)f(x, y) \\ &\leq \frac{1}{4h(t)} \end{aligned}$$

for all $t \geq 0$. Integrating on both sides, we obtain

$$\begin{aligned} x(t) &\leq x_0 + \int_0^t \frac{1}{4(1+h(s))} ds \\ &\leq x_0 + \int_0^\infty \frac{1}{4(1+h(s))} ds \leq \delta \end{aligned}$$

for all $t \geq 0$ where $\delta := x_0 + \int_0^\infty \frac{1}{4h(s)} ds < +\infty$. As a result we obtain that

$$\limsup_{t \rightarrow +\infty} x(t) \leq \delta.$$

Next, we show that

$$\limsup_{t \rightarrow +\infty} y(t) < +\infty.$$

To this end, we consider (6) and obtain

$$\dot{y} \leq b(t)yg(x, y) \leq \mu b(t)y$$

for all $t \geq 0$. Then integrating both sides from 0 to t , we obtain that

$$y(t) \leq y_0 + \mu \int_0^t b(s)y(s)ds.$$

By applying the Gronwall inequality, we conclude that

$$\begin{aligned} y(t) &\leq y_0 e^{\mu \int_0^t b(s)ds} \\ &\leq y_0 e^{\mu \int_0^\infty b(s)ds} \\ &\leq y_0 e^{\mu \lambda} := M \end{aligned}$$

for all $t \geq 0$ and $\lambda = \int_0^\infty b(s)ds < +\infty$. Thus it follows that

$$\limsup_{t \rightarrow +\infty} y(t) \leq M.$$

This completes the proof. In addition, if

$$\lim_{t \rightarrow +\infty} h(t) = +\infty$$

one can easily see that

$$\limsup_{t \rightarrow +\infty} \dot{x}(t) \leq 0.$$

Finally, we note that the conclusion of the theorem holds true independent of the choice of the functions d, k, ρ, a and f .

Theorem 2 .Suppose (x, y) is a solution of the system (*). If $r(y) := \frac{\rho(y)}{y}$ is bounded and

$$\int_0^\infty k(s)ds < +\infty \quad \text{and} \quad \int_0^\infty d(s)ds < +\infty.$$

Then

$$\liminf_{t \rightarrow +\infty} y(t) > 0.$$

Proof:

Let $\beta = \sup_{t \in [0, \infty)} \frac{\rho(y)}{y} < +\infty$ and $L = \int_0^\infty [\beta k(s) + d(s)]ds < +\infty$. Then it follows that

$$\frac{\dot{y}}{y} \geq -d(t) - \beta k(t)$$

for all $t \geq 0$. Integrating both sides from 0 to t , we obtain that

$$\begin{aligned} y(t) &\geq y_0 \exp \left\{ - \int_0^t [\beta k(s) + d(s)] ds \right\} \\ &\geq y_0 \exp \left\{ - \int_0^\infty [\beta k(s) + d(s)] ds \right\} \\ &\geq y_0 \exp \{-L\} \end{aligned}$$

for all $t \geq 0$. Since $y_0 > 0$, we conclude that

$$\liminf_{t \rightarrow +\infty} y(t) > 0.$$

Note again that the theorem is independent of the choice of the functions h, a, b, f, g and γ .

Theorem 3 . Assume that (x, y) is a solution of (*). Suppose

- (i) $\gamma(x) = x^2$ for all $x \in [0, \infty)$.
- (ii) there exists a continuous $\mu : [0, \infty) \rightarrow (0, \infty)$ with $f(x, y) \leq \mu(t)x$ for all t and for all (x, y) such that

$$\int_0^\infty \mu(s) ds := \lambda < +\infty.$$

- (iii) a is nonconstant with $a(t) \geq 1, h(t) \geq 1$ for all t such that

$$\int_0^\infty \frac{h(t)}{a(t)} dt := \theta < +\infty.$$

Then $\liminf_{t \rightarrow +\infty} x(t) > 0$.

Proof: Suppose (x, y) is a solution of the system. Considering equation (*) together with the assumptions, we get that

$$\frac{dx}{dt} \geq -x^2 - a(t)f(x, y) - h(t)x^2$$

for all t , for all x and for all y . Now dividing both sides by x^2 and $a(t)$, we obtain that

$$\frac{\dot{x}}{x^2} \geq \frac{-1}{a(t)} - \frac{f(x, y)}{x^2} - \frac{h(t)}{a(t)}$$

for all t , for all x and for all y . Integrating both sides of the inequality from 0 to t , we get that

$$- \int_0^t \frac{\dot{x}(s)}{x(s)^2} ds \leq \int_0^t \frac{1}{a(s)} ds + \int_0^t \frac{\mu(s)}{x(s)} ds + \int_0^t \frac{h(s)}{a(s)} ds$$

for all $t \geq 0$. Finally, integrating the left hand sides gives the inequality

$$\begin{aligned} \frac{1}{x(t)} &\leq \frac{1}{x_0} + \int_0^t \frac{1}{a(s)} ds + \int_0^t \frac{\mu(s)}{x(s)} ds + \int_0^t \frac{h(s)}{a(s)} ds \\ &\leq \frac{1}{x_0} + \int_0^\infty \frac{1}{a(s)} ds + \int_0^\infty \frac{h(s)}{a(s)} ds + \int_0^t \frac{\mu(s)}{x(s)} ds \\ &\leq \tau + \int_0^t \frac{\mu(s)}{x(s)} ds \end{aligned}$$

for all $t \geq 0$ where

$$0 < \tau := \frac{1}{x_0} + \int_0^\infty \frac{1}{a(s)} ds + \int_0^\infty \frac{h(s)}{a(s)} ds < +\infty.$$

Then by applying the Gronwall inequality, it follows that

$$\begin{aligned} \frac{1}{x(t)} &\leq \tau \exp \left\{ \int_0^t \mu(s) ds \right\} \\ &\leq \tau \exp \left\{ \int_0^\infty \mu(s) ds \right\} \end{aligned}$$

Consequently, it follows that

$$x(t) \geq \frac{1}{\tau e^\lambda}$$

for all $t \geq 0$. Thus

$$\liminf_{t \rightarrow +\infty} x(t) > 0.$$

This completes the proof.

Corollary 1 . *Suppose the assumptions in Theorem 1, Theorem 2 and Theorem 3 are satisfied. Then the system is permanent.*

Corollary 2 . *Consider equation (*) for all $t \geq 0$. If there exists $\varepsilon \in (\frac{1}{a}, 1) \cap (0, 1)$ such that $f(x, y) \geq \varepsilon x$ for all $x > 0$ and for all y , then*

$$\limsup_{t \rightarrow +\infty} x(t) = 0.$$

Proof: From (*), we get that

$$\dot{x} \leq x - a\varepsilon x$$

for all $t \geq 0$. Integrating both sides from 0 to t , we obtain that

$$x(t) \leq x_0 e^{(1-a\varepsilon)t}$$

for all $t \geq 0$. Since $1 - a\varepsilon < 0$, it follows easily that

$$\limsup_{t \rightarrow \infty} x(t) = 0.$$

Consequently, we obtain that

$$\lim_{t \rightarrow +\infty} x(t) = 0.$$

This corollary indicates that the assumption that a is nonconstant can't be omitted from Theorem 3 to prevent the prey from extinction.

In the next section, we consider examples of different predator prey models as an application of the above theorems.

Example 1 : Consider predator prey model non-constant harvesting functions h and k of the prey and predator species as follows:

$$\begin{aligned}\dot{x} &= x(1-x) - a(t)\frac{xy}{y+x} - h(t)x^2 \\ \dot{y} &= y(-d(t) + b(t)\frac{x}{y+x}) - \frac{k(t)y}{c+y}\end{aligned}$$

where initial conditions $x_0 = x(0) > 0$, $y_0 = y(0) > 0$ and $x(t)$ and $y(t)$ represent the densities of the prey and predator species for time t , $a(t)$ is birth rate of the prey and $b(t)$ and $d(t)$ are the birth and death rates of the predator respectively. Let $f(x, y) := \frac{xy}{x+y}$, $g(x, y) = \frac{x}{x+y}$, $\rho(y) = \frac{y}{c+y}$, $x > 0$, $y > 0$. Observe that the functions g and $r(y) = \frac{\rho(y)}{y}$ is bounded. If a, b, d, k and h satisfy the assumptions of the above theorem (for example $h(t) = (t+1)^4, b(t) = d(t) = \frac{1}{(t+1)^2} = k(t)$ and $a(t) = (t+1)^2$ for all $t \geq 0$, then conclusions of theorems 1 and 2 hold.

Biological interpretation: Theorems 1, 2 and 3 imply that the system is permanent. Moreover, it shows precisely that the conditions prevent the species from mutual extinction even if there is harvesting aggressiveness in both species.

Example 2 : Consider the following model with Holling III-type functional response with monotonically increasing harvesting functions as follows.

$$\begin{aligned}\dot{x} &= x(1-x) - a(t)\frac{x^2y}{y^2+x^2} - h(t)x^2 \\ \dot{y} &= y(-d(t) + b(t)\frac{x^2}{y^2+x^2}) - \frac{k(t)y}{c+y}\end{aligned}$$

Taking $f(x, y) = \frac{x^2 y}{x^2 + y^2}$, $g(x, y) = \frac{x^2}{x^2 + y^2}$ and $\rho(y) = \frac{y}{c(t) + y}$, a, b, d and k satisfy the assumptions of the theorem. Then the conclusion follows. The same **biological interpretation** follows as example 1.

Example 3 : Consider a modification of the predator prey model(Gause type) without delay and Michaelis-Menten functional response with harvesting functions of both species given by:

$$\begin{aligned}\dot{x} &= x(1 - x) - a(t)\frac{xy}{1 + x} - h(t)x^2 \\ \dot{y} &= y(-d(t) + b(t)\frac{x}{1 + x}) - \frac{k(t)y}{c + y}\end{aligned}$$

The same conclusion follows if the functions satisfy the assumptions of Theorem 1,2 and 3.

Remark: The above theorems assure that the extinction and survival of the two species is highly influenced by the death, birth and harvesting rates rather than the harvesting and interaction functions in the system. For example, the predator will never extincts even if there is a monotonically increasing non-linear harvesting function ρ and the prey never extincts even though it is harvested more aggressively as compared to the predator. Mutual co-existence is possible under suitable assumptions on the rates. The densities of these populations is mainly controlled by the functions a, b, d, k and h .

In the next section, we use linear analysis to study the stability properties of the equilibrium points and conditions of extinction and survival of the species with rational harvesting of the prey and quadratic harvesting of the predator.

3. Stability analysis of Michaelis -Menten type Systems

In this section, we are interested in analyzing the local stability of equilibria(if exist) of the systems of the form

$$\dot{x} = x(1 - x) - \frac{axy}{y + x} - \frac{hx}{c + x} \quad (1)$$

$$\dot{y} = y(-d + \frac{bx}{y + x}) - K(y) \quad (2)$$

where $K(y) = ky$ and $K(y) = ky^2$ and a, b, c, d, h and k are all positive constants. we are interested to find equilibrium points of this system $\in [0, \infty) \times [0, \infty)$ because solutions outside this set are not biologically interpretable. Analysis of the stability of these equilibrium points will be given. To this

end, we prove the following theorems. If $K(y) = ky$, we get the following system

$$\begin{aligned} \dot{x} &= x(1-x) - \frac{axy}{y+x} - \frac{hx}{c+x} \\ \dot{y} &= y(-e + \frac{bx}{y+x}) \end{aligned}$$

where $e := d + k$ with all the constants are positive. The stability properties of this system is discussed in [1] with e considered as d in [1]. Now taking $K(y) = ky^2$, we get the following results.

Theorem 4 :Consider the system (1) and (2). Let $\beta = \frac{(1-c)^2}{4} + c - h$, $\gamma = \frac{\beta d^2}{4(a-\beta)k(k+d-b)}$ and $\delta = \frac{k\gamma^2}{b-k\gamma}$. Then every equilibrium point (x, y) in R_+^2 lies in $(0, \delta] \times (0, \gamma]$ provided that $k\gamma < b < k + d$, $ac > \beta$ and $h < \frac{(1+c)^2}{4}$.

Proof: Let (x, y) be an equilibrium point of the system, i.e. (x, y) is a solution of the following system

$$x(1-x) - \frac{axy}{x+y} - \frac{hx}{c+x} = 0 \tag{3}$$

$$y(-d + \frac{bx}{x+y}) - ky^2 = 0 \tag{4}$$

in $(0, \alpha] \times (0, \gamma]$. Assume that $x > 0$ and $y > 0$. Dividing equation (4) by y and solving for y in terms of x , we get that

$$y = \frac{-d + \sqrt{d^2 - 4k(k+d-b)x}}{2k} > 0. \tag{5}$$

Now, from equation (3), it follows that

$$\begin{aligned} \frac{ay}{x+y} &= (1-x) - \frac{h}{c+x} \\ &= \frac{(1-x)(c+x) - h}{c+x} \\ &= \frac{-x^2 + (1-c)x + c - h}{c+x} \\ &\leq \frac{\frac{(1-c)^2}{4} + (c-h)}{c+x} \\ &\leq \frac{(1-c)^2}{4c} + \frac{c-h}{c} := \frac{\beta}{c} \end{aligned}$$

Since $h < \frac{(1+c)^2}{4}$, it follows that $\beta := \frac{(1-c)^2}{4} + (c - h) > 0$. From equation (5), it follows that $y > 0$ if

$$x \leq \frac{d^2}{4k(k + d - b)} := \delta$$

and solving for y from the inequality, we obtain that

$$y \leq \frac{\beta}{(ac - \beta)} \left[\frac{d^2}{4k(k + d - b)} \right] = \gamma.$$

On the other hand, one can easily see that $\gamma > 0$ provided that $ac > \beta$ and $k + d > b$. Consequently, we conclude that (x, y) is an equilibrium point in $(0, \delta] \times (0, \gamma]$. This proves the theorem.

Corollary 3 : Consider the system (1) and (2). Then

- (i) there is no equilibrium point on the positive Y axis.
- (ii) $A = (x_1, 0)$ is an equilibrium point in R_+^2 ; provided that $0 < c \leq 1$ and $h < \frac{(c+1)^2}{4}$;
- (iii) $B = (x_2, 0)$ is an equilibrium point in R_+^2 provided that $0 < c < 1$ and $c < h < \frac{(c+1)^2}{4}$.

where

$$x_i = \frac{(1 - c) + (-1)^{i+1} \sqrt{(1 - c)^2 + 4(c - h)}}{2}, i = 1, 2.$$

Under the assumptions of the above theorem and corollary, the system can have at most three equilibrium points in R_+^2 . At the equilibrium points A and B the prey survives and the predator becomes extinct. The survival of the prey depends on the harvesting effort h and the capturing rate c .

Theorem 5 . Consider the system (1) and (2). If $h \leq \frac{1+2c}{4}$ and $0 < c \leq 1$, then A is an equilibrium point in R_+^2 and

- (i) A is a saddle point if $b > d$.
- (ii) A is asymptotically stable if $b < d$.

Proof of (i):

The Jacobian matrix of the system at any (x, y) is given by

$$J(x, y) = \begin{pmatrix} 1 - 2x - \frac{ay^2}{(x+y)^2} - \frac{hc}{(x+c)^2} & \frac{-ax^2}{(x+y)^2} \\ \frac{by^2}{(x+y)^2} & -d - 2ky + \frac{bx^2}{(x+y)^2} \end{pmatrix}$$

Then the Jacobian at A is given by

$$J(x_1, 0) = \begin{pmatrix} c - \sqrt{(1+c)^2 - 4h} - \frac{4hc}{(1+c+\sqrt{(1+c)^2 - 4h})^2} & -a \\ 0 & -d + b \end{pmatrix}$$

Let δ_A be the determinant of $J(x_1, 0)$. Since $h < \frac{1+2c}{4}$ and $b > d$, it follows that

$$c - \sqrt{(1+c)^2 - 4h} < 0$$

and finally we get that $\delta_A < 0$. Thus A is saddle point.

Proof of (ii) It is easy to see that all the eigenvalues of $J(x_1, 0)$ are negative if $b < d$. Thus A is asymptotically stable.

Theorem 6 Consider the system (1) and (2). If $\frac{2c+1}{4} < h \leq \frac{(1+c)^2}{4}$, $c < h$ and $0 < c \leq 1$, then B is an equilibrium point in R_+^2 and

(i) is saddle point if $b < d$.

(ii) is a stable node or stable focus if $b > d$.

Proof of (i):

The Jacobian at B is given by

$$J(x_2, 0) = \begin{pmatrix} c + \sqrt{(1+c)^2 - 4h} - \frac{4hc}{(1+c - \sqrt{(1+c)^2 - 4h})^2} & -a \\ 0 & -d + b \end{pmatrix}$$

Let δ_A be the determinant of $J(x_1, 0)$ and δ_B be the determinant at $J(x_2, 0)$. First observe that

$$1 + c - \sqrt{(1+c)^2 - 4h} > 0.$$

Otherwise, we get $h \leq 0$ which is not the case. Now we verify that

$$c + \sqrt{(1+c)^2 - 4h} > \frac{4hc}{(1+c - \sqrt{(1+c)^2 - 4h})}. \tag{6}$$

Suppose the inequality is not true. It is clear from the hypothesis that $c - \sqrt{(1+c)^2 - 4h} \geq 0$. It is easy to see that we get that $h < \frac{2c+1}{4}$ which is impossible by the hypothesis. Thus the inequality (6) is true. Therefore, $\delta_B < 0$ if $b < d$. Hence B is a saddle point which proves (i). The proof for (ii) follows similarly.

From theorem 6, it is easy to see that mutual coexistence is impossible, i.e. the predator extincts in both cases and the prey survives and its existence depends on h, c, b and d . Moreover, all the trajectories starting at (x_0, y_0) will approach the point B if $b > d$.

Theorem 7 . Suppose (x, y) is an equilibrium point of (1) and (2). Let $\sigma := \max\{\delta, \gamma\}$. If

$$h \leq \frac{4\sigma^2(1 - 2\delta - a - 2b)}{c} \quad \text{and} \quad 2\delta + a + 2b < 1, \tag{7}$$

then (x, y) is unstable.

Proof: Let λ be an eigenvalue of

$$J(x, y) := \begin{pmatrix} 1 - 2x - \frac{ay^2}{(x+y)^2} - \frac{hc}{(x+y)^2} & \frac{-ax^2}{(x+y)^2} \\ \frac{by^2}{(x+y)^2} & -d - 2ky + \frac{bx^2}{(x+y)^2} \end{pmatrix}.$$

Consider the function g defined by

$$\begin{aligned} g(\lambda) &= \left(-d - 2ky + \frac{bx^2}{(x+y)^2} - \lambda\right) \left(1 - 2x - \frac{ay^2}{(x+y)^2} - \frac{hc}{(x+y)^2} - \lambda\right) \\ &+ \frac{abx^2b^2}{(x+y)^4}. \end{aligned}$$

Then it is easy to see that $g(1) > 0$. Moreover observe that $\lambda_* := \frac{bx^2}{(x+y)^2} \leq b < 1$ and

$$g(\lambda_*) = (-d - 2ky) \left(1 - 2x - \frac{ay^2}{(x+y)^2} - \frac{hc}{(x+y)^2} - \lambda_*\right).$$

On the other hand,

$$1 - 2x - \frac{ay^2}{(x+y)^2} - \frac{hc}{(x+y)^2} - \lambda_* \geq 1 - 2\delta - a - 2b - \frac{hc}{(x+y)^2} > 0$$

provided that

$$h < \frac{4\sigma^2(1 - 2\delta - a - 2b)}{c} \quad \text{and} \quad 2\delta + a + 2b < 1.$$

Consequently, we obtain that $g(\lambda_*) < 0$. Thus there exists $\lambda_0 \in (\lambda_*, 1)$ such that $g(\lambda_0) = 0$. This proves that $J(x, y)$ has a positive eigenvalue. Therefore the equilibrium point (x, y) is unstable.

4. Discussion

In section 2, we considered a generalized time dependent nonlinear ordinary differential equation and proved that the solution of the system is bounded, i.e. the system is permanent. Permanence of different predator prey models are discussed which extends previous research from autonomous to time dependent models which extends the study of such systems from limited type of harvesting functions to more general time dependent rates. As a result mutual coexistence is possible for any time t . It is proved that the densities of the species is controlled by these rates. Since these rates are practically nonconstant for any time in real life condition, the model is better than those with constant rates. Prediction for possible extinction and coexistence of the species is easier

than that of stability analysis of the equilibrium points. In section 3, we have extended the study from rational harvesting of the prey to harvesting of both species with rational harvesting of the prey and quadratic harvesting of the predator. The result in section 2 has an advantage over the stability analysis because in the latter case the solution trajectory may fail to be bounded for future time.

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