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Asymptotics of Orthogonal Polynomials on a System of Complex Arcs and Curves: The Case of a Measure with Denumerable Set of Mass Points off the System

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Abstract

We study the strong asymptotics of orthogonal polynomials with respect to a measure of the type $\sigma = \alpha + \gamma$, where α denotes a positive Szegö measure on the system of arcs and curves E in the complex plane and γ is a discrete measure concentrated on an infinite number of mass points in the region exterior to the system E. Our main result is the explicit strong asymptotic formulas for the corresponding orthogonal polynomials.

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1 Introduction

The study of orthogonal polynomials over a closed rectifiable curve of the complex plane was introduced about 88 years ago by Szegő in his paper [18], and later continued by Szegő himself and other authors as Smirnov, Keldysh, Lavrentiev, Korovkin, Suetin and Geronimous. Orthogonal polynomials over several arcs and curves have also been studied by Akhiezer [1], Aptekarev [2], Peherstorfer and Yuditskii [15], Widom [19], and for an orthogonality measure with denumerable set of masses point outside the curve or arc, by Khaldi [10], [11], Gonchar [5], Kaliaguine [7], [8], Li and Pan [13].

In Widom [19], we find some general results concerning the strong asymptotics for orthogonal polynomials with respect to measure supported by a finite system of arcs and curves and which satisfy Szegő condition. An extension of Widom's results has been given by Kaliaguine and Kononova [9] for a measure concentrated on a system of arcs and curves and perturbed by finite Blaschke sequence of point masses.

In this paper, we study the strong asymptotics problem for the monic orthogonal polynomials with respect to the finite positive Borel measure $\sigma = \alpha + \gamma$, with an infinite compact support in the complex plane, where α denotes the absolutely continuous part of the measure σ on E *i.e.*,

$$d\alpha(\xi) = \rho(\xi)\,|d\xi|\,,\ \rho \geq 0,\ \rho \in L^1(E, |d\xi|)\,, \tag{1.1}$$

and γ is a point measure supported on a denumerable set of points $\{z_k\}_{k=1}^{\infty}$ off the system *i.e.*,

$$\gamma = \sum_{k=1}^{\infty} A_k \delta_{z_k},\ A_k > 0,\ \sum_{k=1}^{\infty} A_k < \infty. \tag{1.2}$$

Denote by $\{T_n(z)\}$ the monic polynomial of degree n orthogonal with respect to the measure σ *i.e.*,

$$T_n(z) = z^n + ...,\ \int_E T_n(\xi).\overline{\xi}^p \rho(\xi)\,|d\xi| + \sum_{k=1}^{\infty} A_k T_n(z_k).\overline{z_k}^p = 0, \quad p = 0, 1, ..., n-1.$$

Our investigation is based on the extremal property for the polynomials $T_n(z)$. One of the main problems of research in the study of orthogonal polynomials is to find an asymptotics representation of $T_n(z)$ as $n \to \infty$. Other commonly used names are power asymptotic, Szegő asymptotic or full exterior asymptotic.

We try to bring over some of the foremost ideas of Kaliaguine and Kononova [9] to the infinite case, where the situation turns out to be much more difficult, for this reason, no much is known about orthogonal polynomials on a system of curves and arcs, although they play an important role in modern solid state physics see Gasper and Cyrot [4], Heine [6], since the densities of states live on several arcs.

We give in section 2 some basic definitions and notations to be able to state our results, we also recall the definition of the complex Green function, Szegő function, and describe the Hardy multi-valued functions. We then expose the extremal problems in section 3. Our main results, namely theorem 4.3, theorem 4.4 are proved in section 4.

2 Definitions and notations

Definition 2.1 *A rectifiable curve or an arc E is said to be of class C^{2+} if in the canonical parameterisation $z(s)$ of E, the second derivative of the function $z(s)$ satisfies a Lipschitz condition with some positive exponent.*

Let $E = \bigcup_{k=1}^{p} E_k$ be a union of complex rectifiable Jordan curves and arcs of class C^{2+} with $E_j \cap E_k = \emptyset,\ j \neq k$. By $E^{(1)}$ we denote the union of curves in E and by $E^{(2)}$ the union of arcs.

Denote by Ω the connected component of $\mathbb{C} \setminus E$ such that $\infty \in \Omega$. Let $\rho(\xi) \geq 0$ be a weight function on E with $\int_E \rho(\xi)\,|d\xi| < +\infty$.

The boundary $\partial\Omega$ of the region Ω is essentially the set E where any arc of E is taken twice. In what follows we use the following notation:

$$\oint_E f(\xi)\,|d\xi| = \int_{\partial\Omega} f(\xi)\,|d\xi|.$$

Definition 2.2 *Let $g(z, z_0)$ be the real Green function for Ω with singularity at z_0, that is:*

1. *$g(z, z_0)$ is harmonic in $\Omega \backslash \{z_0\}$.*
2. *Function $[g(z, z_0) - \log(1/|z - z_0|)]$ is harmonic in the neighborhood of z_0. For $z_0 = \infty$ the function $[g(z) - \log|z|]$ is harmonic at ∞.*
3. *$\lim_{z\to\xi} g(z, z_0) = 0$ for $\xi \in \partial\Omega$ a.e. (nontangential limit).*

Denote by $\breve{g}(z, z_0)$ it's harmonic conjugate, then the function $G(z, z_0) = g(z, z_0) + i\breve{g}(z, z_0)$ is called complex Green function for Ω with the pole at z_0. We denote by $g(z)$ and $G(z)$ real and complex Green functions with singularity at infinite $(z_0 = \infty)$. The function $\Phi(z) = \exp[G(z)]$ is locally analytic in Ω, has no zero there with a pole at infinity. For the boundary values $\Phi(z)$ on $\partial\Omega$ we have $|\Phi(\xi)|_{\xi\in\partial\Omega} = 1$. If $p > 1$, the function $\Phi(z)$ is multi-valued in Ω. More precisely, $|\Phi(z)| = \exp(g(z))$ is single-valued and $\arg\Phi(z)$ is multi-valued. According to Caratheodory theorem [3] the complex derivative $\Phi'(z)$ has nontangential limit value almost everywhere on E.

Definition 2.3 *The logarithmic capacity of the set E is the positive number $C(E) = \exp(-\gamma)$, where γ is the so-called Robin's constant for $\Omega : \gamma = \lim_{z\to\infty} [g(z) - \log|z|]$. For the case $p = 1$ one has $C(E) = \lim_{z\to\infty} [z/\Phi(z)]$.*

The weight function ρ satisfies the Szegő condition on E if

$$\oint_E \log\rho(\xi)\,|\Phi'(\xi)|\,|d\xi| > -\infty. \tag{2.1}$$

Definition 2.4 *Suppose the Szegő condition (2.1) is satisfied for the weight function $\rho(\xi)$. Then there exists the real function $h(z)$ harmonic in Ω with* *the boundary condition on* E : $h(\xi)|_{\xi\in\partial\Omega} = \log(\rho(\xi))$. *The function* $R(z) = \exp\left[h(z) + i\tilde{h}(z)\right]$ *is locally analytic in* Ω, *has a nontangential limit values on* $\partial\Omega$ *and* $|R(z)|_{\partial\Omega} = \rho(\xi)$. *The function* $D(z) = \sqrt{R(z)} = \exp\left[\frac{1}{2}\left(h(z) + i\tilde{h}(z)\right)\right]$ *is called Szegő function associated with the weight function* $\rho(\xi)$.

Definition 2.5 *The Hardy space* $H^2(\Omega,\rho)$ *of multi-valued functions is the space of functions* f *locally analytic in* Ω *with single-valued modulus and multi-valued argument such that the function* $\left|f(z)^2 R(z)\right|$ *has a harmonic majorant in* Ω. *Each function* f *from* $H^2(\Omega,\rho)$ *admits a nontangential limit values a.e. on* E *and*

$$\|f\|^2_{H^2(\Omega,\rho)} = \oint_E |f(\xi)|^2 \rho(\xi)|d\xi| < \infty. \tag{2.2}$$

Definition 2.5 *We define the Szegő kernel function* $K_n(z,z_0)$ *in the space* $H^2(\Omega,\rho)$ *with reproducing property: for any function* $f \in H^2(\Omega,\rho)$ *one has*

$$\oint_E f(\xi)\overline{K_n(\xi,z_0)}\rho(\xi)|d\xi| = f(z_0). \tag{2.3}$$

3 Extremal problems

We denote by P_n be the set of polynomials of degree n, and $P_{n,1}$ the set of monic polynomials of degree n.

Define $m_n(\sigma)$, $\mu(\sigma)$, $m_n(\sigma_N)$, $\mu(\sigma_N)$, $m_n(\alpha)$ and $\mu(\alpha)$ the extremal values of the following problem, respectively,

$$m_n(\sigma) = \min\left\{\int_E |Q_n(\xi)|^2 \rho(\xi)|d\xi| + \sum_{k=1}^{\infty} A_k |Q_n(z_k)|^2,\ Q_n \in P_{n,1}\right\}, \tag{3.1}$$

$$\mu(\sigma) = \inf\left\{\|\varphi\|^2_{H^2(\Omega,\rho)} : \varphi \in H^2(\Omega,\rho),\ |\varphi(\infty)| = 1,\ \varphi(z_k) = 0,\ k = 1,2,...\right\}, \tag{3.2}$$

$$m_n(\sigma_N) = \min\left\{\int_E |Q_n(\xi)|^2 \rho(\xi)|d\xi| + \sum_{k=1}^{N} A_k |Q_n(z_k)|^2,\ Q_n \in P_{n,1}\right\}, \tag{3.3}$$

$$\mu(\sigma_N) = \inf\left\{\|\varphi\|^2_{H^2(\Omega,\rho)} : \varphi \in H^2(\Omega,\rho),\ |\varphi(\infty)| = 1,\ \varphi(z_k) = 0,\ k = 1,2,...,N\right\}, \tag{3.4}$$

where the measure

$$\sigma_N = \alpha + \sum_{k=1}^{N} A_k \delta_{z_k}, \tag{3.5}$$

$$m_n(\alpha) = \min \left\{ \int_E |Q_n(\xi)|^2 \rho(\xi) |d\xi|, \; Q_n \in P_{n,1} \right\}, \tag{3.6}$$

$$\mu(\alpha) = \inf \left\{ \|\varphi\|^2_{H^2(\Omega,\rho)} : \varphi \in H^2(\Omega, \rho), \; |\varphi(\infty)| = 1 \right\}. \tag{3.7}$$

We denote, respectively, by ϕ^∞, ϕ^N and φ^* the extremal functions of the problems (3.2), (3.4) and (3.7). Finally we also have $T_n^N(z)$ and $T_n(z)$ are, respectively, the optimal solutions of the extremal problems (3.3) and (3.1).

It is proved in [19] that the extremal function of the problem (3.7) is unique up to the complex constant factor of modulus 1.

Lemma 3.1 *The extremal function ϕ^∞ of the problem (3.2) is given by $\phi^\infty(z) = \varphi^*(z) . B(z)$; in addition,*

$$\mu(\sigma) = \mu(\alpha) \prod_{k=1}^{\infty} |\Phi(z_k)|^2,$$

where the function B is the product

$$B(z) = \prod_{k=1}^{\infty} \frac{\Phi(\infty, z_k)}{\Phi(z, z_k)}.$$

Proof. The proof is the same as given in [10].

Remark 3.2 *Function $B(z)$ is analytic in Ω and has the following properties*

$$B(z_k) = 0; \; |B(\infty)| = 1; \; |B(\xi)|_{\xi\in\Omega} = \prod_{k=1}^{\infty} |\Phi(z_k)|.$$

In the case when Ω is simply $(p = 1)$ connected that is E is a curve or an arc the function $B(z)$ is the Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\overline{\Phi(z_k)} - 1} \frac{\left|\Phi(z_k)^2\right|}{\Phi(z_k)}.$$

4 Main results

In this section we obtain the strong asymptotics for the polynomials $\{T_n(z)\}_{n\in\mathbb{N}}$, for $z \in K$, for all compact $K \subset \Omega\backslash \{z_k\}_{k=1}^{\infty}$.

Definition 4.1 *A measure $\sigma = \alpha + \gamma$ is said to belong to a class A, if the absolutely continuous part α and the discrete part γ satisfy the conditions (1.1), (1.2) and the Blaschke's condition, i.e.,*

$$\sum_{k=1}^{\infty} \left(\Phi\left(z_k\right) - 1\right) < \infty. \tag{4.1}$$

Remark 4.2 *The condition (4.1) is natural and it guarantees the convergence of the Blaschke product*

$$\prod_{k=1}^{\infty} \left|\Phi\left(z_k\right)\right|^2.$$

Theorem 4.3 *Let $\sigma = \alpha + \sum\limits_{k=1}^{\infty} A_k \delta_z$ be a measure which belongs to A; then*

$$\lim_{N\to\infty} m_n\left(\sigma_N\right) = m_n\left(\sigma\right)$$

Proof. The extremal property of $T_n^N(z)$, gives

$$m_n\left(\sigma_N\right) \le \int\limits_E \left|T_n\left(\xi\right)\right|^2 \rho\left(\xi\right)\left|d\xi\right| + \sum_{k=1}^{N} A_k \left|T_n\left(z_k\right)\right|^2 \le m_n\left(\sigma\right),$$

then

$$m_n\left(\sigma_N\right) \le m_n\left(\sigma\right). \tag{4.2}$$

On the other hand, the extremal property of $T_n(z)$, implies that

$$m_n\left(\sigma\right) \le \int\limits_E \left|T_n^N\left(\xi\right)\right|^2 \rho\left(\xi\right)\left|d\xi\right| + \sum_{k=1}^{\infty} A_k \left|T_n^N\left(z_k\right)\right|^2 = m_n\left(\sigma_N\right) + \sum_{k=N+1}^{\infty} A_k \left|T_n^N\left(z_k\right)\right|^2. \tag{4.3}$$

According to the reproducing property of the kernel function $K_n(\xi, z)$, and $T_n^N(z) \in P_n$, we have

$$T_n^N\left(z_k\right) = \oint\limits_E T_n^N\left(\xi\right) \overline{K_n\left(\xi, z_k\right)} \rho\left(\xi\right)\left|d\xi\right|.$$

The Schwarz inequality and the fact that $K_n(z, z_k) \in P_n$ imply

$$\left|T_n^N(z_k)\right|^2 \le \int_E \left|T_n^N(\xi)\right|^2 \rho(\xi)\,|d\xi| \,.\oint_E |K_n(\xi, z_k)|^2 \rho(\xi)\,|d\xi| \le m_n(\sigma_N)\,.K_n(z_k, z_k)\,. \tag{4.4}$$

The inequalities (1.2), (4.3) and (4.4) imply

$$m_n(\sigma) \le m_n(\sigma_N) + \sum_{k=N+1}^{\infty} A_k m_n(\sigma_N)\,.K_n(z_k, z_k) \le m_n(\sigma_N)\left[1 + \sup_{k \ge N+1} K_n(z_k, z_k) \sum_{k=N+1}^{\infty} A_k\right],$$

so we have

$$\frac{m_n(\sigma)}{m_n(\sigma_N)} \le 1 + \delta_N \text{ where } \delta_N = \sup_{k \ge N+1} K_n(z_k, z_k) \sum_{k=N+1}^{\infty} A_k \underset{N \to \infty}{\to} 0. \tag{4.5}$$

Using (4.2) and (4.5) we obtain

$$m_n(\sigma) \le \liminf_{N \to \infty} m_n(\sigma_N) \le \limsup_{N \to \infty} m_n(\sigma_N) \le m_n(\sigma)\,,\ \forall n,$$

this implies that

$$\lim_{N \to \infty} m_n(\sigma_N) = m_n(\sigma) \quad \forall n.$$

Theorem 4.4 *Let E be a system of curves and arcs from the class C^{2+} and let $\sigma = \alpha + \gamma$ be a measure which belong to a class A. In addition, for all n and N,*

$$m_n(\sigma_N) \le \left(\prod_{k=1}^{N} |\Phi(z_k)|\right)^2 m_n(\alpha)\,. \tag{4.6}$$

Then we have

$$1)\ \lim_{n \to \infty} \frac{m_n(\sigma)}{[C(E)]^{2n}} = \mu(\sigma)\,,$$

$$2)\ \lim_{n \to \infty} \int_E \left|\frac{T_n(\xi)}{C(E)^n} - \Psi(\xi)\right|^2 \rho(\xi)\,|d\xi| = 0,$$

$$3)\ T_n(z) = [C(E)\,\Phi(z)]^n\,.\,[\phi^{\infty}(z) + \varepsilon_n(z)]\,,$$

where $\varepsilon_n \to 0$ *uniformly on the compact subsets of* Ω *and*

$$\Psi(\xi) = \begin{cases} \Phi^n(\xi)\,\phi^\infty(\xi) & \xi \in E^{(1)} \\ \Phi_+^n(\xi)\,\phi_+^\infty(\xi) + \Phi_-^n(\xi)\,\phi_-^\infty(\xi) & \xi \in E^{(2)} \end{cases}.$$

Proof. We start with the proof of

$$\limsup_{n\to\infty} \frac{m_n(\sigma)}{[C(E)]^{2n}} \le \mu(\sigma). \tag{4.7}$$

By passing to the limit when N tends to infinity in (4.6) and using theorem 4.3, we obtain

$$\frac{m_n(\sigma)}{[C(E)]^{2n}} \le \left(\prod_{k=1}^{\infty} |\Phi(z_k)|^2\right) \frac{m_n(\alpha)}{[C(E)]^{2n}}. \tag{4.8}$$

On the other hand it is proved in ([19], theorem 12.3) that

$$\lim_{n\to\infty} \frac{m_n(\alpha)}{[C(E)]^{2n}} = \mu(\alpha). \tag{4.9}$$

Using (4.8), (4.9) and lemma 3.1, we get

$$\limsup_{n\to\infty} \frac{m_n(\sigma)}{[C(E)]^{2n}} \le \left(\prod_{k=1}^{\infty} |\Phi(z_k)|^2\right) \mu(\alpha) = \mu(\sigma). \tag{4.10}$$

Now show that

$$\liminf_{n\to\infty} \frac{m_n(\sigma)}{[C(E)]^{2n}} \ge \mu(\sigma).$$

Consider the integral

$$I_n = \int_E \left| \frac{T_n(\xi)}{[C(E)]^n} - \Psi(\xi) \right|^2 \rho(\xi)\,|d\xi|,$$

and transform it into three integrals and evaluate each of them

$$\begin{aligned} I_n &= \int_E \left|\frac{T_n(\xi)}{[C(E)]^n}\right|^2 \rho(\xi)\,|d\xi| + \int_E |\Psi(\xi)|^2 \rho(\xi)\,|d\xi| - 2\mathrm{Re}\int_E \frac{T_n(\xi)}{[C(E)]^n}\overline{\Psi(\xi)}\rho(\xi)\,|d\xi|, \\ &= I_n^{(1)} + I_n^{(2)} + I_n^{(3)}. \end{aligned}$$

Integral $I_n^{(1)}$: From the definition of $m_n(\sigma)$ we get

$$I_n^{(1)} = \int_E \left|\frac{T_n(\xi)}{[C(E)]^n}\right|^2 \rho(\xi)\,|d\xi| = \frac{m_n(\sigma)}{C(E)^{2n}} - \sum_{k=1}^{\infty} A_k \left|\frac{T_n(z_k)}{C(E)^n}\right|^2 \le \frac{m_n(\sigma)}{C(E)^{2n}}. \tag{4.11}$$

Integral $I_n^{(2)}$: For the curves we have

$$\int\limits_{E^{(1)}} |\Psi(\xi)|^2 \rho(\xi)\, |d\xi| = \oint\limits_{E^{(1)}} |\phi^\infty(\xi)|^2 \rho(\xi)\, |d\xi|\,.$$

For the arcs we have

$$\int\limits_{E^{(2)}} |\Psi(\xi)|^2 \rho(\xi)\, |d\xi| = \oint\limits_{E^{(2)}} |\phi^\infty(\xi)|^2 \rho(\xi)\, |d\xi| + 2\mathrm{Re}\int\limits_{E} \overline{\Phi_+^n(\xi)\, \phi_+^\infty(\xi)}\Phi_-^n(\xi)\, \phi_-^\infty(\xi)\, \rho(\xi)\, |d\xi|\,.$$

The second integral tends to zero as $n \to \infty$ (lemma 12.1 of [19]). Finally we obtain

$$I_n^{(2)} = \mu(\sigma) + \alpha_n, \quad \alpha_n \to 0. \tag{4.12}$$

Integral $I_n^{(3)}$: Since $\overline{\Phi_+} = \dfrac{1}{\Phi_+}$ we have

$$2\mathrm{Re}\int\limits_{E} \frac{T_n(\xi)}{[C(E)]^n}\overline{\Psi(\xi)}\rho(\xi)\, |d\xi| = 2\mathrm{Re}\int\limits_{E} \frac{T_n(\xi)}{[C(E)\, \Phi^n(\xi)]^n}\overline{\phi^\infty}\rho(\xi)\, |d\xi|\,,$$

then by proceeding as in [9] we get

$$I_n^{(3)} = 2\mu(\sigma) + \beta_n, \quad \beta_n \to 0. \tag{4.13}$$

Using (4.11), (4.12) and (4.13) we obtain

$$0 \le I_n \le \frac{m_n(\sigma)}{C(E)^{2n}} + \mu(\sigma) + \alpha_n - 2\mu(\sigma) - \beta_n. \tag{4.14}$$

this implies

$$\liminf_{n\to\infty} \frac{m_n(\sigma)}{[C(E)]^{2n}} \ge \mu(\sigma)\,. \tag{4.15}$$

The inequalities (4.10) and (4.15) prove (1) of the theorem 4.4.

On the other hand, by passing to the limit when n tends to infinity in (4.14) and taking into account (1) of theorem we get

$$0 \le \lim_{n\to\infty} I_n \le \lim_{n\to\infty}\left[\frac{m_n(\sigma)}{[C(E)]^{2n}} - \mu(\sigma) + \alpha_n - \beta_n\right] = 0.$$

It prove (2) of theorem 4.4.

For the proof of the third part of the theorem 4.4 we use the reproducing kernel $K_n(\xi,z)$, We have

$$\begin{aligned}
\frac{T_n(z)}{C(E)^n\Phi^n(z)} &= \oint_E \frac{T_n(\xi)}{C(E)^n\Phi^n(\xi)}\overline{K_n(\xi,z)}\rho(\xi)\,|d\xi| = \\
&= \int_{E^{(1)}} C(E)^{-n}T_n(\xi)\,\Phi^{-n}(\xi)\,\overline{K_n(\xi,z)}\rho(\xi)\,|d\xi| + \\
&\quad + \int_{E^{(2)}} C(E)^{-n}T_n(\xi)\left(\Phi_+^{-n}(\xi)\,\overline{K_{n+}(\xi,z)} + \Phi_-^{-n}(\xi)\,\overline{K_{n-}(\xi,z)}\right)\rho(\xi)\,|d\xi| \\
&= \int_{E^{(1)}} \left(C(E)^{-n}T_n(\xi) - \Psi(\xi)\right).\Phi^{-n}(\xi)\,\overline{K_n(\xi,z)}\rho(\xi)\,|d\xi| + \\
&\quad + \int_{E^{(2)}} \left(C(E)^{-n}T_n(\xi) - \Psi(\xi)\right)\left(\Phi_+^{-n}(\xi)\,\overline{K_{n+}(\xi,z)} + \Phi_-^{-n}(\xi)\,\overline{K_{n-}(\xi,z)}\right)\rho(\xi)\,|d\xi| + \\
&\quad + \int_{E^{(1)}} \Psi(\xi)\,\Phi^{-n}(\xi)\,\overline{K_n(\xi,z)}\rho(\xi)\,|d\xi| + \\
&\quad + \int_{E^{(2)}} \Psi(\xi)\left(\Phi_+^{-n}(\xi)\,\overline{K_{n+}(\xi,z)} + \Phi_-^{-n}(\xi)\,\overline{K_{n-}(\xi,z)}\right)\rho(\xi)\,|d\xi|,
\end{aligned}$$

Two first integrals go to zero (part 2 of the theorem). We transform the last integral:

$$\begin{aligned}
&\int_{E^{(2)}} \Psi(\xi)\left(\Phi_+^{-n}(\xi)\,\overline{K_{n+}(\xi,z)} + \Phi_-^{-n}(\xi)\,\overline{K_{n-}(\xi,z)}\right)\rho(\xi)\,|d\xi| \\
= &\int_{E^{(2)}} \left(\Phi_+^{n}(\xi)\,\phi_+^{\infty}(\xi) + \Phi_-^{n}(\xi)\,\phi_-^{\infty}(\xi)\right)\left(\Phi_+^{-n}(\xi)\,\overline{K_{n+}(\xi,z)} + \Phi_-^{-n}(\xi)\,\overline{K_{n-}(\xi,z)}\right)\rho(\xi)\,|d\xi| \\
= &\int_{E^{(2)}} \phi^{\infty}(\xi)\,\overline{K_n(\xi,z)}\rho(\xi)\,|d\xi| + \\
&+ \int_{E^{(2)}} \left(\Phi_+^{n}(\xi)\,\phi_+^{\infty}(\xi)\,\Phi_-^{-n}(\xi)\,\overline{K_{n-}(\xi,z)} + \Phi_-^{n}(\xi)\,\phi_-^{\infty}(\xi)\,\Phi_+^{-n}(\xi)\,\overline{K_{n+}(\xi,z)}\right)\rho(\xi)\,|d\xi|.
\end{aligned}$$

The second integral tends to zero (lemma 12.1 of [19]). Finally

$$\frac{T_n(z)}{C(E)^n\Phi^n(z)} = \oint_{E^{(1)}\cup E^{(2)}} \phi^{\infty}(\xi)\,\overline{K_n(\xi,z)}\rho(\xi)\,|d\xi| = \phi^{\infty}(z) + \theta_n,$$

where $\theta_n \to 0$. This completes the proof of the theorem 4.4.

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