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A Related Fixed Point Theorem on Four Metric Spaces

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Abstract

A fixed point theorem on four metric spaces is proved. This theorem gereralizes and extends the result obtained in [1] from three metric spaces to four metric spaces. Further, a generalization of the Theorem 1 proved in [1] is obtained as corollary of our theorem.

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1 Introduction

In [1], the following fixed point theorem is proved:

Theorem 1.1 *Let* $(X, d_1), (Y, d_2)$ *and* (Z, d_3) *be complete metric spaces. Let* $T : X \to Y, S : Y \to Z, R : Z \to X$ *be three mappings satisfying the following inequalities:*

$$\begin{aligned} d_1(RSy, RSTx) &\leq \frac{cf_1(x,y)}{g_1(x,y)} \\ d_2(TRz, TRSy) &\leq \frac{cf_2(y,z)}{g_2(y,z)} \\ d_3(STx, STRz) &\leq \frac{cf_3(z,x)}{g_3(z,x)} \end{aligned}$$

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for all $x \in X, y \in Y$ *and* $z \in Z$ *for which* $g_1(x,y) \neq 0, g_2(y,z) \neq 0, g_3(z,x) \neq 0$, *where* $0 \leq c < 1$ *and*

$$\begin{aligned}
f_1(x,y) &= \max\{d_1(x,RSTx)d_3(Sy,STx), d_1(x,RSTx)d_2(y,TRSy), d_1(x,RSy)d_2(y,Tx)\}\\
f_2(y,z) &= \max\{d_2(y,TRSy)d_1(Rz,RSy), d_2(y,TRSy)d_3(z,STRz), d_2(y,TRz)d_3(z,Sy)\}\\
f_3(z,x) &= \max\{d_3(z,STRz)d_2(Tx,TRz), d_3(z,STRz)d_1(x,RSTx), d_3(z,STx)d_1(x,Rz)\}\\
g_1(x,y) &= \max\{d_1(x,RSy), d_1(x,RSTx), d_2(Tx,TRSy)\}\\
g_2(y,z) &= \max\{d_2(y,TRz), d_2(y,TRSy), d_3(Sy,STRz)\}\\
g_3(z,x) &= \max\{d_3(z,STx), d_3(z,STRz), d_1(Rz,RSTx)\}
\end{aligned}$$

then RST *has a unique fixed point* $\alpha \in X$, TRS *has a unique fixed point* $\beta \in Y$ *and* STR *has a unique fixed point* $\gamma \in Z$. *Further,* $T\alpha = \beta, S\beta = \gamma$ *and* $R\gamma = \alpha$.

In this paper we obtain a new result (cf. Theorem 2.1) concerning fixed points for mappings on four metric spaces. We generalize and extend the result of Theorem 1.1 from three metric spaces to four metric spaces. As a corollary of our theorem we obtain a new theorem which generalizes the Theorem 1.1.

2 Main results

Now, we will give and prove our theorem as follows:

Theorem 2.1 *Let* $(X,d_1),(Y,d_2)$, (Z,d_3) *and* (U,d_4) *be complete metric spaces. Let* $T : X \to Y, S : Y \to Z, R : Z \to U$ *and* $Q : U \to X$ *be four mappings satisfying the following inequalities:*

$$d_1(QRSy,QRSTx) \leq \frac{cF_1(x,y)}{G_1(x,y)} \qquad (1)$$

$$d_2(TQRz,TQRSy) \leq \frac{cF_2(y,z)}{G_2(y,z)} \qquad (2)$$

$$d_3(STQu,STQRz) \leq \frac{cF_3(z,u)}{G_3(z,u)} \qquad (3)$$

$$d_4(RSTx,RSTQu) \leq \frac{cF_4(u,x)}{G_4(u,x)} \qquad (4)$$

for all $x \in X, y \in Y$, $z \in Z$ *and* $u \in U$ *for which* $G_1(x,y) \neq 0, G_2(y,z) \neq$

$0, G_3(z,u) \neq 0, G_4(u,x) \neq 0$, where $0 \leq c < 1$ and

$$\begin{aligned}
F_1(x,y) &= \max \ \{d_1(x,QRSTx)d_4(RSy,RSTx); d_1(x,QRSTx)d_3(Sy,STx);\\
&\qquad d_1(x,QRSTx)d_2(y,TQRSy); d_1x,QRSy)d_2(y,Tx)\}\\
F_2(y,z) &= \max \ \{d_2(y,TQRSy)d_1(QRz,QRSy); d_2(y,TQRSy)d_4(Rz,RSy);\\
&\qquad d_2(y,TQRSy)d_3(z,STQRz); d_2(y,TQRz)d_3(z,Sy)\}\\
F_3(z,x) &= \max \ \{d_3(z,STQRz)d_2(TQu,TQRz); d_3(z,STQRz)d_1(Qu,QRz);\\
&\qquad d_3(z,STQRz)d_4(u,RSTQu); d_3(z,STQu)d_4(u,Rz)\}\\
F_4(u,x) &= \max \ \{d_4(u,RSTQu)d_3(STx,STQu); d_4(u,RSTQu)d_2(Tx,TQu);\\
&\qquad d_4(u,RSTQu)d_1(x,QRSTx); d_4(u,RSTx)d_1(x,Qu)\}\\
G_1(x,y) &= \max \ \{d_1(x,QRSy), d_1(x,QRSTx), d_2(Tx,TQRSy)\}\\
G_2(y,z) &= \max \ \{d_2(y,TQRz), d_2(y,TQRSy), d_3(Sy,STQRz)\}\\
G_3(z,u) &= \max \ \{d_3(z,STQu), d_3(z,STQRz), d_4(Rz,RSTQu)\}\\
G_4(u,x) &= \max \ \{d_4(u,RSTx), d_4(u,RSTQu), d_1(Qu,QRSTx)\}
\end{aligned}$$

then $QRST$ has a unique fixed point $\alpha \in X$, $TQRS$ has a unique fixed point $\beta \in Y$, $STQR$ has a unique fixed point $\gamma \in Z$ and $RSTQ$ has a unique fixed point $\delta \in U$. Further, $T\alpha = \beta, S\beta = \gamma, R\gamma = \delta$ and $Q\delta = \alpha$.

Proof. Let $x_0 \in X$ be an arbitrary point. We define the four sequences $(x_n), (y_n), (z_n)$ and (u_n) with X, Y, Z and U respectively as follows:

$$x_n = (QRST)^n x_0; y_n = Tx_{n-1}; z_n = Sy_n; u_n = Rz_n$$

for all $n \in \mathbb{N}$. We assume that $x_n \neq x_{n+1}, y_n \neq y_{n+1}, z_n \neq z_{n+1}, u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$. Otherwise, if $x_n = x_{n+1}$ for some n, then $y_{n+1} = y_{n+2}, z_{n+1} = z_{n+2}, u_{n+1} = u_{n+2}$ and we could put $x_n = \alpha, y_{n+1} = \beta, z_{n+1} = \gamma$ and $u_{n+1} = \delta$. If $y_n = y_{n+1}$, then $z_n = z_{n+1}, u_n = u_{n+1}$ and the later equalities imply that $x_n = x_{n+1}$. Simliarly, if $z_n = z_{n+1}$ or $u_n = u_{n+1}$, then again $x_n = x_{n+1}$.

Taking $z = z_{n-1}$ and $y = y_n$ in (2), we obtain:

$$\begin{aligned}
d_2(y_n,y_{n+1}) &= d_2(TQRz_{n-1},TQRSy_n) \leq \frac{cF_2(y_n,z_{n-1})}{G_2(y_n,z_{n-1})} =\\
&= \frac{c\max\{d_2(y_n,y_{n+1})d_1(x_{n-1},x_n); d_2(y_n,y_{n+1})d_4(u_{n-1},u_n);}{\max\{d_2(y_n,y_n); d_2(y_n,y_{n+1}); d_3(z_n,z_n)\}}\\
&\quad \frac{d_2(y_n,y_{n+1})d_3(z_{n-1},z_n); d_2(y_n,y_n)d_3(z_{n-1},z_n)\}}{\max\{d_2(y_n,y_n); d_2(y_n,y_{n+1}); d_3(z_n,z_n)\}} =\\
&= \frac{c\max\{d_2(y_n,y_{n+1})[d_1(x_{n-1},x_n,d_4(u_{n-1},u_n),d_3(z_{n-1},z_n)]\}}{d_2(y_n,y_{n+1})}\\
&= c\max\{d_1(x_{n-1},x_n), d_3(z_{n-1},z_n), d_4(u_{n-1},u_n)\} \qquad (5)
\end{aligned}$$

Taking $u = u_{n-1}$ and $z = z_n$ in (3) we obtain:

$$\begin{aligned}
d_3(z_n, z_{n+1}) &= d_3(STQu_{n-1}, STQRz_n) \leq \frac{cF_3(z_n, u_{n-1})}{G_3(z_n, u_{n-1})} = \\
&= \frac{c \max\{d_3(z_n, z_{n+1})d_2(y_n, y_{n+1}); d_3(z_n, z_{n+1})d_1(x_{n-1}, x_n);}{\max\{d_3(z_n, z_n); d_3(z_n, z_{n+1}); d_3(u_n, u_n)\}} \\
&\quad \frac{d_3(z_n, z_{n+1})d_4(u_{n-1}, u_n); d_3(z_n, z_n)d_4(u_{n-1}, u_n)\}}{\max\{d_3(z_n, z_n); d_3(z_n, z_{n+1}); d_3(u_n, u_n)\}} = \\
&= \frac{c \max\{d_2(z_n, z_{n+1})[d_2(y_n, y_{n+1}, d_1(x_{n-1}, x_n), d_4(u_{n-1}, u_n)]\}}{d_3(z_n, z_{n+1})} \\
&= c \max\{d_2(y_n, y_{n+1}), d_1(x_{n-1}, x_n), d_4(u_{n-1}, u_n)\}
\end{aligned}$$

and using (5) we obtain:

$$\begin{aligned}
d_3(z_n, z_{n+1} &\leq c \max \ \{cd_1(x_{n-1}, x_n), cd_3(z_{n-1}, z_n), cd_4(u_{n-1}, u_n), d_1(x_{n-1}, x_n), \\
&\qquad d_4(u_{n-1}, u_n)\} \\
&= c \max \ \{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n), d_4(u_{n-1}, u_n)\} \qquad (6)
\end{aligned}$$

Taking $u = u_n$ and $x = x_n$ in (4) we obtain:

$$\begin{aligned}
d_4(u_n, u_{n+1}) &= d_4(RSTx_{n-1}, RSTQu_n) \leq \frac{cF_4(u, x)}{G_4(u, x)} = \\
&= \frac{c \max\{d_4(u_n, u_{n+1})d_3(z_n, z_{n+1}); d_4(u_n, u_{n+1})d_2(y_n, y_{n+1});}{\max\{d_4(u_n, u_n); d_4(u_n, u_{n+1}); d_1(x_n, x_n)\}} \\
&\quad \frac{d_4(u_n, u_{n+1})d_1(x_{n-1}, x_n); d_4(u_n, u_n)d_1(x_{n-1}, x_n)\}}{\max\{d_4(u_n, u_n); d_4(u_n, u_{n+1}); d_1(x_n, x_n)\}} = \\
&= \frac{c \max\{d_4(u_n, u_{n+1})[d_3(z_n, z_{n+1}, d_2(y_n, y_{n+1}), d_1(x_{n-1}, x_n)]\}}{d_4(u_n, u_{n+1})} \\
&= c \max\{d_3(z_n, z_{n+1}), d_2(y_n, y_{n+1}), d_1(x_{n-1}, x_n)\}
\end{aligned}$$

and using (5) and (6) we obtain:

$$d_4(u_n, u_{n+1}) \leq c \max\{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n), d_4(u_{n-1}, u_n)\} \tag{7}$$

Taking $x = x_n$ and $y = y_n$ in (1) we obtain:

$$\begin{aligned}
d_1(x_n, x_{n+1}) &= d_1(QRSy_n, QRSTx_n) \leq \frac{cF_1(x_n, y_n)}{G_1(x, y)} = \\
&= \frac{c \max\{d_1(x_n, x_{n+1})d_4(u_n, u_{n+1}); d_1(x_n, x_{n+1})d_3(z_n, z_{n+1});}{\max\{d_1(x_n, x_n); d_1(x_n, x_{n+1}); d_2(y_{n+1}, y_{n+1})\}} \\
&\quad \frac{d_1(x_n, x_{n+1})d_2(y_n, y_{n+1}); d_1(x_n, x_n)d_2(y_n, y_{n+1})\}}{\max\{d_1(x_n, x_n); d_1(x_n, x_{n+1}); d_2(y_{n+1}, y_{n+1})\}} = \\
&= \frac{c \max\{d_1(x_n, x_{n+1})[d_4(u_n, u_{n+1}, d_3(z_n, z_{n+1}), d_2(y_n, y_{n+1})]\}}{d_1(x_n, x_{n+1})} \\
&= c \max\{d_4(u_n, u_{n+1}), d_3(z_n, z_{n+1}), d_2(y_n, y_{n+1})\}
\end{aligned}$$

and using (5), (6) and (7) we obtain:

$$d_1(x_n, x_{n+1}) \leq c \max\{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n) d_4(u_{n-1}, u_n)\}. \tag{8}$$

It now follows by induction on using inequalities (5), (6), (7) and (8) that:

$$\begin{aligned}
d_1(x_n, x_{n+1}) &\leq c^{n-1} \max\{d_1(x_1, x_2); d_3(z_1; z_2), d_4(u_1, u_2)\} \\
d_2(y_n, y_{n+1}) &\leq c^{n_1} \max\{d_1(x_1, x_2); d_3(z_1; z_2), d_4(u_1, u_2)\} \\
d_3(z_n, z_{n+1}) &\leq c^{n-1} \max\{d_1(x_1, x_2); d_3(z_1; z_2), d_4(u_1, u_2)\} \\
d_4(u_n, u_{n+1}) &\leq c^{n-1} \max\{d_1(x_1, x_2); d_3(z_1; z_2), d_4(u_1, u_2)\}.
\end{aligned}$$

Since $0 \leq c < 1$, the sequences $(x_n), (y_n), (z_n)$ and (u_n) are Cauchy sequences. Since $(X, d_1), (Y, d_2)$, (Z, d_3) and (U, d_4) are complete metric spaces, we have:

$$\lim_{n\to\infty} x_n = \alpha \in X; \lim_{n\to\infty} y_n = \beta \in Y; \lim_{n\to\infty} z_n = \gamma \in Z; \lim_{n\to\infty} u_n = \delta \in U.$$

Taking $x = x_n$ and $y = \beta$ in (1) we obtain:

$$\begin{aligned}
d_1(QRS\beta, x_{n+1}) &= d_1(QRS\beta, QRSTx_n) \leq \frac{cF_1(x_n, \beta)}{G_1(x_n, \beta)} = \\
&= \frac{c \max\{d_1(x_n, x_{n+1}) d_4(RS\beta, u_{n+1}); d_1(x_n, x_{n+1}) d_3(S\beta, z_{n+1});}{\max\{d_1(x_n, QRS\beta); d_1(x_n, x_{n+1}); d_2(y_{n+1}, TQRS\beta)\}} \\
&\quad \frac{d_1(x_n, x_{n+1}) d_2(\beta, TRSQ\beta); d_1(x_n, QRS\beta) d_2(\beta, y_{n+1})\}}{\max\{d_1(x_n, QRS\beta); d_1(x_n, x_{n+1}); d_2(y_{n+1}, TQRS\beta)\}}.
\end{aligned}$$

Letting n tending in infinity we get

$$d_1(QRS\beta, \alpha) \leq 0$$

from which it follows

$$QRS\beta = \alpha.$$

In the same way, using the inequalities (2), (3) and (4) it can be shown that

$$RST\alpha = \delta, STQ\delta = \gamma, TQR\gamma = \beta.$$

Taking $y = y_n$ and $z = S\beta$ in (2) we obtain:

$$\begin{aligned}
d_2(TQRS\beta, y_{n+1}) &= d_2(TQRS\beta, TQRSy_n) \leq \frac{cF_2(y_n, S\beta)}{G_2(y_n, S\beta)} = \\
&= \frac{c \max\{d_2(y_n, y_{n+1}) d_1(QRS\beta, x_n); d_2(y_n, y_{n+1}) d_4(RS\beta, u_n);}{\max\{d_2(y_n, TQRS\beta); d_2(y_n, y_{n+1}); d_3(z_n, STQRS\beta)\}} \\
&\quad \frac{d_2(y_n, y_{n+1}) d_3(S\beta, STQRS\beta); d_2(y_n, TQRS\beta) d_3(S\beta, z_n)\}}{\max\{d_2(y_n, TQRS\beta); d_2(y_n, y_{n+1}); d_3(z_n, STQRS\beta)\}}.
\end{aligned}$$

Letting n tending in infinity and since $QRS\beta = \alpha$ we get

$$d_2(TQRS\beta, \beta) \leq \frac{cd_2(\beta, TQRS\beta)d_3(S\beta, \delta)}{\max\{d_2(\beta, TQRS\beta), d_3(\gamma, ST\alpha)\}}.$$

We distinguish here two cases:
If $\max\{d_2(\beta, TQRS\beta), d_3(\gamma, ST\alpha)\} = d_2(\beta, TQRS\beta)$ we have

$$d_2(TQRS\beta, \beta) = d_2(T\alpha, \beta) \leq \frac{c \max\{d_2(\beta, TQRS\beta)d_3(S\beta, \delta)\}}{d_2(\beta, TQRS\beta)\}} = cd_3(S\beta, \gamma).$$

If $\max\{d_2(\beta, TQRS\beta), d_3(\gamma, ST\alpha)\} = d_3(\gamma, ST\alpha)$ and $d_2(\beta, TQRS\beta) \neq 0$ we have

$$\begin{aligned} d_2(TQRS\beta, \beta) &= d_2(T\alpha, \beta) \leq \frac{c \max\{d_2(\beta, TQRS\beta)d_3(S\beta, \gamma)\}}{d_3(\gamma, ST\alpha)\}} \leq \\ &\leq \frac{cd_2(\beta, TQRS\beta)d_3(S\beta, \gamma)}{d_2(\beta, TQRS\beta)} = cd_3(S\beta, \gamma). \end{aligned}$$

Thus, we always have:

$$d_2(TQRS\beta, \beta) = d_2(T\alpha, \beta) \leq cd_3(S\beta, \gamma). \tag{9}$$

In the same way, taking $z = z_n$ and $u = R\gamma$ using (3) we obtain:

$$\begin{aligned} d_3(STQR\gamma, z_{n+1}) &= d_3(STQR\gamma, STQRz_n) \leq \frac{cF_3(z_n, R\gamma)}{G_3(z_n, R\gamma)} = \\ &= \frac{c \max\{d_3(z_n, z_{n+1})d_2(TQR\gamma, y_{n+1}); d_3(z_n, z_{n+1})d_1(QR\gamma, x_n);}{\max\{d_3(z_n, STQR\gamma); d_3(z_n, z_{n+1}); d_4(u_n, RSTQR\gamma)\}} \\ &\quad \frac{d_3(z_n, z_{n+1})d_4(R\gamma, RSTQR\gamma); d_3(z_n, STQR\gamma)d_4(R\gamma, u_n)\}}{\max\{d_3(z_n, STQR\gamma); d_3(z_n, z_{n+1}); d_4(u_n, RSTQR\gamma)\}}. \end{aligned}$$

Letting n tending in infinity and since $TQR\gamma = \beta$ we get

$$d_3(STQR\gamma, \gamma) = d_3(S\beta, \gamma) \leq \frac{cd_3(\gamma, STQR\gamma)d_4(R\gamma, \gamma)}{\max\{d_3(\gamma, STQR\gamma), d_4(\delta, RS\beta)\}} \leq cd_4(R\gamma, \gamma).$$

Thus,

$$d_3(STQR\gamma, \gamma) = d_3(S\beta, \gamma) \leq cd_4(R\gamma, \gamma). \tag{10}$$

In the same way, using the inequalities (4) and (5) it can be shown that

$$d_4(RSTQ\delta, \delta) = d_4(R\gamma, \delta) \leq cd_1(Q\delta, \alpha) \tag{11}$$

and

$$d_1(QRST\alpha, \alpha) = d_1(Q\delta, \alpha) \le cd_2(T\alpha, \beta) \tag{12}$$

Using (9), (10), (11) and (12) we obtain:

$$\begin{aligned} d_2(TQRS\beta, \beta) &= d_2(T\alpha, \beta) \le cd_3(S\beta, \gamma) \le c^2 d_4(R\gamma, \gamma) \le c^3 d_1(Q\delta, \alpha) \le \\ &\le c^4 d_2(T\alpha, \beta) = c^4 d_2(TQRS\beta, \beta) \end{aligned}$$

from which it follows

$$TQRS\beta = \beta; T\alpha = \beta; S\beta = \gamma; R\gamma = \delta; Q\delta = \alpha$$

since $0 \le c < 1$.

It is also obvious that

$$STQR\gamma = \gamma; RSTQ\delta = \delta; QRST\alpha = \alpha.$$

Therefore, α, β, γ and δ are fixed points of $QRST, TQRS, STQR$ and $RSTQ$ respectively.

Now let we show their unicity.

Let assume now that α' is another fixed point of $QRST$ different from α. Using the inequality (1) for $y = T\alpha$ and $x = \alpha'$ we get

$$\begin{aligned} d_1(\alpha, \alpha') &= d_1(QRST\alpha, QRST\alpha') \le \\ &\le \frac{c \max\{d_1(\alpha', \alpha')d_4(RST\alpha, RST\alpha'); d_1(\alpha', \alpha')d_3(ST\alpha, ST\alpha');}{\max\{d_1(\alpha', \alpha); d_1(\alpha', \alpha'); d_2(T\alpha', T\alpha)\}} \\ &\quad \frac{d_1(\alpha', \alpha')d_2(T\alpha, TQRST\alpha); d_1(\alpha', \alpha')d_2(T\alpha, T\alpha')\}}{\max\{d_1(\alpha', \alpha); d_1(\alpha', \alpha'); d_2(T\alpha', T\alpha)\}} \le \\ &\le \frac{cd_1(\alpha', \alpha)d_2(T\alpha, T\alpha')}{\max\{d_1(\alpha, \alpha'); d_2(T\alpha', T\alpha)\}}. \end{aligned}$$

If $\max\{d_1(\alpha, \alpha'); d_2(T\alpha', T\alpha)\} = d_2(T\alpha', T\alpha)$, then we get $d_1(\alpha, \alpha') \le cd_1(\alpha', \alpha)$ from which it follows $\alpha = \alpha'$.

If $\max\{d_1(\alpha, \alpha'); d_2(T\alpha', T\alpha)\} = d(\alpha, \alpha')$, then

$$d_1(\alpha, \alpha') \le cd_2(T\alpha, T\alpha'). \tag{13}$$

In the same way, taking $z = ST\alpha$ and $y = T\alpha'$ in (2) we obtain:

$$\begin{aligned} d_2(T\alpha, T\alpha') &= d_2(TQRST\alpha, TQRST\alpha') \le \\ &\le \frac{\alpha \max\{d_2(T\alpha', TQRST\alpha')d_1(\alpha, \alpha'); d_2(T\alpha', T\alpha')d_4(RST\alpha, RST\alpha');}{\max\{d_2(T\alpha', T\alpha); d_2(T\alpha', T\alpha'); d_3(ST\alpha', ST\alpha)\}} \\ &\quad \frac{d_2(T\alpha', T\alpha')d_3(ST\alpha, STQRST\alpha); d_2(T\alpha', T\alpha)d_3(ST\alpha, ST\alpha')\}}{\max\{d_2(T\alpha', T\alpha); d_2(T\alpha', T\alpha'); d_3(ST\alpha', ST\alpha)\}} = \\ &= \frac{cd_2(T\alpha', T\alpha)d_3(ST\alpha, ST\alpha')}{\max\{d_2(T\alpha', T\alpha); d_2(T\alpha', T\alpha'); d_3(ST\alpha', ST\alpha)\}} \le cd_3(ST\alpha', ST\alpha). \end{aligned}$$

Thus,

$$d_2(T\alpha, T\alpha') \le cd_3(ST\alpha', ST\alpha). \tag{14}$$

By (3), for $u = RST\alpha$ and $z = ST\alpha'$ we get

$$d_3(ST\alpha, ST\alpha') \le cd_4(RST\alpha', RST\alpha). \tag{15}$$

By (4), for $x = QRST\alpha$ and $u = RST\alpha'$ we get

$$d_4(RST\alpha, RST\alpha') \le cd_1(QRST\alpha', QRST\alpha) = cd_1(\alpha', \alpha). \tag{16}$$

By (13), (14), (15) and (16) it follows

$$d_1(\alpha, \alpha') < cd_2(T\alpha, T\alpha') \le c^2 d_3(ST\alpha, ST\alpha') \le c^3 d_4(RST\alpha, RST\alpha') \le c^4 d_1(\alpha, \alpha')$$

which is impossible, since $0 \le c < 1$.

Thus, α is a unique fixed point of $QRST$.

In the same way it can be proved that β is the unique fixed point of $TQRS$, γ is the unique fixed point of $STQR$ and δ is the unique fixed point of $RSTQ$.

This completes the proof of the theorem.

Corollary 2.2 *If we consider in Theorem 2.1 the metric space (U, d_4) the same with the metric space (X, d_1), (that is $U = X, d_4 = d_1$), and the mapping Q as the identity mapping of X ($Qx = x, \forall x \in X$), then we obtain a fixed point theorem on three metric spaces which is a generalization of Theorem 1[1].*

Proof. We apply the inequalities (1), (2), (3) and (4) for $d_1 = d, U = X, u = x$ and $Q = I$ as the identity mapping.

The inequality (1) takes the form:

$$d_1(RSy, TRSy) \le \tfrac{cF_1(x,y)}{G_1(x,y)} \tag{1'}$$

where

$$\begin{aligned} F_1(x,y) &= \max \{d_1(x, RSTx)d_1(RSy, RSTx); d_1(x, RSTx)d_3(Sy, STx); \\ &\qquad d_1(x, RSTx)d_2(y, TRSy); d_1(x, RSy)d_2(y, Tx)\} \ge \\ &\ge \max \{d_1(x, RSTx)d_3(Sy, STx); d_1(x, RSTx)d_2(y, TRSy); \\ &\qquad d_1(x, RSy)d_2(y, Tx)\} = f_1(x,y) \end{aligned}$$

and $G_1(x,y) = \max\{d_1(x, RSy); d_1(x, RSTx); d_2(Tx, TRSy)\} = g_1(x,y)$.

The inequality (2) takes the form:

$$d_2(TRz, TRSy) \le \tfrac{cF_2(y,z)}{G_2(y,z)} \tag{2'}$$

where

$$\begin{aligned} F_2(y,z) &= \max \ \{d_2(y,TRSy)d_1(Rz,RSy); d_2(y,TRSy)d_1(Rz,RSy); \\ &\quad d_2(y,TRSy)d_3(z,STRz); d_2(y,TRz)d_3(z,Sy)\} \geq \\ &\geq \max \ \{d_2(y,TRSy)d_1(Rz,RSy); d_2(y,TRSy)d_3(z,STRz); \\ &\quad d_2(y,TRz)d_3(z,Sy)\} = f_1(y,z) \end{aligned}$$

and $G_2(y,z) = \max\{d_2(y,TRz); d_2(y,TRSy); d_3(Sy,STRz)\} = g_2(y,z)$.

The inequality (3) takes the form:

$$d_3(STx, STRz) \leq \tfrac{cF_3(z,x)}{G_3(z,x)} \qquad (3')$$

where

$$\begin{aligned} F_3(z,x) &= \max \ \{d_3(z,STRz)d_2(Tx,TRz); d_3(z,STRz)d_1(x,Rz); \\ &\quad d_3(z,STRz)d_1(x,RSTx); d_3(z,STx)d_1(x,Rz)\} \geq \\ &\geq \max \ \{d_3(z,STRz)d_2(Tx,TRz); d_3(z,STRz)d_1(x,RSTx); \\ &\quad d_3(z,STx)d_1(x,Rz)\} = f_3(z,x) \end{aligned}$$

and $G_3(z,x) = \max\{d_3(z,STx); d_2(z,STRz); d_1(Rz,RSTx)\} = g_3(z,x)$.

The inequality (4) takes the form:

$$d_4(RSTx, RSTx) \leq \tfrac{cF_4(x,x)}{G_4(x,x)} \qquad (4')$$

which holds true for all $x \in X$ since the left side is zero.

Thus, in case of three mappings (special case $Q = I$), the satisfying of conditions (1), (2), (3), (4) of Theorem 2.1 is reduced in satisfying of the following conditions:

$$\begin{aligned} d_1(RSy,TRSy) &\leq \frac{cF_1(x,y)}{G_1(x,y)} && (1') \\ d_2(TRz,TRSy) &\leq \frac{cF_2(y,z)}{G_2(y,z)} && (2') \\ d_3(STx,STRz) &\leq \frac{cF_3(z,x)}{G_3(z,x)} && (3') \end{aligned}$$

and, from the above it follows whenever the conditions

$$\begin{aligned} d_1(RSy,RSTx) &\leq \frac{cf_1(x,y)}{g_1(x,y)} \\ d_2(TRz,TRSy) &\leq \frac{cf_2(y,z)}{g_2(y,z)} \\ d_3(STx,STRz) &\leq \frac{cf_3(z,x)}{g_3(z,x)} \end{aligned}$$

of Theorem 1 [1] are satisfied, the conditions (1'), (2'), (3') of the special case of Theorem 2.1 are satisfied. Therefore, we obtain a generalization of Theorem 1 [1].

References

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