On Graded \( *G_q \)-Rings

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Abstract

We introduce the conditions \((^\ast S_q)\) and \((^\ast G_q)\) that are the graded versions of the classical conditions \((S_q)\) and \((G_q)\), \(q > 0\) integer. In particular the \(^\ast G_q\)-rings are characterized.

Mathematics Subject Classification: 13A02, 13C15, 13H10, 13A30

Keywords: Graded rings, \((^\ast G_q)\)-condition

1 Introduction

Let \( R \) be a commutative noetherian ring. In [11] Serre’s condition \((S_q)\) and Ischebeck-Auslander’s condition \((G_q)\) are considered and some properties connected to them are studied. If \( R \) is a graded ring, we say that similar properties to the not graded case can be established and studied. In particular the problem started in [2]. Definitions and results are inspired by the results known in the not graded case, however it is interesting to establish them in the graduate case. In fact in the graded algebras it can be determined good properties, as in the local case. It follows the utility.

In [7], [8] \((S_q)\) and \((G_q)\) properties are investigated in the graduate case introducing the conditions \((^\ast S_q)\) and \((^\ast G_q)\). The aim of this paper is to state for graded rings some theoretical properties using the conditions \((^\ast S_q)\) and \((^\ast G_q)\) studied in [3] and [10]. Gorenstein homogeneous sequences are examined and the behavior of the condition \((^\ast G_q)\) in the passage from \(R/(x)\) to \(R\), where is \( R \) is a graded ring and \( x \) is a regular homogeneous element belonging to the homogeneous radical of \( R \), is studied.

The paper is organized as follows. In the section 1 we consider graded noetherian rings that satisfy the Ischebeck-Auslander’s condition, that are said \(^\ast G_q\)-rings and we give a characterization for them. More precisely we give a sufficient condition for a graded ring to be a \(^\ast G_q\)-ring in terms of Gorenstein homogenous sequences, that we call \(^\ast\)G-sequences. Moreover for a \(^\ast S_q\)-ring we
study the conditions \( q \)-Gorenstein and \( q \)-Bass-Ischisawa in the graduate case. In the not graded case, we have the connection between \( q \)-Gorenstein and \( q \)-Bass-Ischisawa rings, more precisely the condition \( q \)-Gorenstein implies the \( q \)-Bass-Ischisawa condition ([5], [12]). In general this implication can not be reversed. In [10] it is proved an equivalence between \( q \)-Gorenstein and \( q \)-Bass-Ischisawa rings if all the maximal ideals have height \( \geq q \). We state the result for a graded ring \( R \) satisfying the same hypotheses on the graded maximal ideals of \( R \).

In section 2 it is studied the behavior of the \((^*S_{q})\) and \((^*G_{q})\) properties under some changes of rings proving in which cases the properties are preserved after a reformulation of statements of [3]. In particular we study these properties in the passage from \( R/(x) \) to \( R \), where \( x \) is a homogeneous regular element of \( R \). We prove that if \((^*S_{q})\) (resp.\((^*G_{q})\)) holds for \( R/(x) \), then \((^*S_{q})\) (resp.\((^*G_{q})\)) holds for \( R \).

2 Preliminary notes

We introduce some preliminary notions.

Let \( R \) be a commutative noetherian graded ring and \( \wp \subset R \) be a prime ideal (not necessarily graded). For us, \( \wp^* \) is the graded ideal of \( R \) generated by all the homogeneous elements of \( \wp \). The ideal \( \wp^* \) is the largest graded ideal contained in \( \wp \) and it is also a prime ideal ([2], 1.5.6). It is clearly that \( \wp = \wp^* \) if \( \wp \) is a graded ideal.

The following definitions are the graded versions of the classical definitions of the conditions \((S_{q})\) and \((G_{q})\), \( q > 0 \) integer ([9], [11]).

**Definition 2.1** Let \( R \) be a commutative noetherian graded ring, \( q \geq 0 \) be an integer. \( R \) satisfies Serre’s condition \((^*S_{q})\) (or is a *\( S_{q} \)-ring) if for all prime ideal \( \wp \) of \( R \)

\[
\text{depth}_{R_{\wp^*}} \geq \min\{q, \dim_{R_{\wp^*}}\}.
\]

**Definition 2.2** \( R \) be a commutative noetherian graded ring and \( q \geq 0 \) be an integer. \( R \) satisfies the \((^*G_{q})\) condition of Ischebeck-Auslander (or is a *\( G_{q} \)-ring) if:

1) \( R \) satisfies the condition \((^*S_{q})\);
2) for all prime ideal \( \wp \) of \( R \) such that \( \dim_{R_{\wp^*}} < q \), \( R_{\wp^*} \) is a Gorenstein ring.

The \((G_{q})\) condition is connected to the Gorenstein sequences introduced in [10]. In such a scheme of things we give a criterion to prove that a ring is a *\( G_{q} \)-ring in connection to *Gorenstein sequences.
Definition 2.3 Let $R$ be a commutative noetherian graded ring. An ordered sequences $x_1, \ldots, x_n$ of non invertible homogeneous elements of $R$ is a $^*G$-sequence or a $^{*G}$-sequence if the following conditions hold:
1) $x_1, \ldots, x_n$ is a homogeneous regular sequence;
2) for every $i \in \{1, \ldots, n\}$ the homogeneous ideal $(x_1, \ldots, x_i)$ has irreducible minimal primary components.

Theorem 2.4 Let $R$ be a commutative noetherian graded ring, $q \geq 0$ be an integer. If every prime ideal $\wp$ of $R$ contains a homogeneous regular sequence of length $\geq \min\{q, \dim R_{\wp^*}\}$, then $R$ is a $^{*S_q}$-ring.

Proof: If we denote $\text{depth}(\wp^*, R)$ the length of any maximal homogeneous sequence on $R$ which is contained in $\wp$, then by hypothesis $\text{depth}(R, \wp^*) \geq \min\{q, \dim R_{\wp^*}\}$, that is each prime ideal $\wp$ of $R$ contains a homogeneous regular sequence of length $\geq \min\{q, \dim R_{\wp^*}\}$.

Moreover for all prime ideal $\wp$ of $R$ such that $\dim R_{\wp^*} = \text{ht}(\wp^*) < q$, one has $n \geq \dim R_{\wp^*}$. By hypothesis any $^*G$-sequence of length $\geq \min\{q, \dim R_{\wp^*}\}$ is a homogeneous regular sequence, that is equivalent to say $\text{depth} R_{\wp^*} \geq \min\{q, \dim R_{\wp^*}\}$, for all $\wp \in \text{Spec}(R)$, by Proposition 2.4. Hence ($^{*S_q}$) property holds.

Theorem 2.5 Let $R$ be a commutative noetherian graded ring and $q \geq 0$ be an integer. If every prime ideal $\wp$ of $R$ contains a $^*G$-sequence of length $\geq \min\{q, \dim R_{\wp^*}\}$, then $R$ is a $^{*G_q}$-ring.

Proof: Suppose that every prime ideal $\wp$ of $R$ contains a $^*G$-sequence $x_1, \ldots, x_n$ of length $n \geq \min\{q, \dim R_{\wp^*}\}$. By hypothesis any $^*G$-sequence of length $\geq \min\{q, \dim R_{\wp^*}\}$ is a homogeneous regular sequence, that is equivalent to say $\text{depth} R_{\wp^*} \geq \min\{q, \dim R_{\wp^*}\}$, for all $\wp \in \text{Spec}(R)$, by Proposition 2.4. Hence ($^*G_q$) property holds.

Moreover for all prime ideal $\wp$ of $R$ such that $\dim R_{\wp^*} = \text{ht}(\wp^*) < q$, one has $n \geq \dim R_{\wp^*}$. By hypothesis for every graded minimal prime ideal $\wp_i \in \text{Ass}(R/(x_1, \ldots, x_i))$, for $i = 1, \ldots, n$, $R_{\wp_i}$ is Gorenstein and $\text{ht}(\wp_i) < q$ ([10]). In particular $\wp^*$ is one of the graded ideals $\wp_i$, hence $R_{\wp^*}$ is Gorenstein for all prime ideal $\wp$ of $R$ such that $\dim R_{\wp^*} < q$. Hence $R$ is a $^{*G_q}$-ring.

Example 2.6 a) Let $R = K[X_1, \ldots, X_n]$ be a graded ring with standard graduation. $R$ is a $^{*G_q}$-ring for all $q \geq 0$ because it is a regular ring, hence $R$ is a Gorenstein graded ring.

b) Let $R = K[X_1, \ldots, X_n]$ and $\{X_1, \ldots, X_n\}$ be a regular sequence. Then by [4] (Cor. 21.19) $S = R/(X_1, \ldots, X_n)$ is a Gorenstein graded ring, that is $S$ is a $^{*G_q}$-ring for all $q \geq 0$.

c) Let $R = K[X_1, X_2, X_3]/(X_1^2, X_1X_2, X_1X_3) = K[x_1, x_2, x_3]$, $x_1^2 = x_1x_2 = x_1x_3 = 0$. $A = R_{(x_1, x_2, x_3)}$ is a local ring with depth $A = 0$. In fact the only regular sequence of $A$ is the empty set and the zero ideal that it generates has
its minimal primary component irreducible. Hence every homogeneous regular sequence of \( A \) is a \(^q\)-G-sequence, in particular every regular sequence of length \( \leq 1 \) is a \(^q\)-G-sequence, that is \( A \) is a \(^*G_0\)-ring. But \( A \) is not a \(^*G_1\)-ring (respectively not Gorenstein). In fact the condition \((^*S_1)\) is not satisfied.

d) Let \( R = K[X_1, X_2, X_3, X_4]/(X_1X_3, X_1X_4, X_2X_3, X_2X_4) = K[x_1, x_2, x_3, x_4] \), \( x_1 x_3 = x_1 x_4 = x_2 x_3 = x_2 x_4 = 0 \). \( A = R(x_1, x_2, x_3, x_4) \) is a local ring with depth \( A = 1 \), that is every homogeneous regular sequence of \( A \) is a \(^q\)-G-sequence, in particular every homogeneous regular sequence of length \( \leq 2 \) is a \(^q\)-sequence, that is \( A \) is a \(^*G_1\)-ring. But \( A \) is not a \(^*G_2\)-ring (respectively not Gorenstein). In fact the condition \((^*S_2)\) is not satisfied.

These examples are the graded versions of those studied in ([10]).

We have the following characterization for \(^*G_q\)-rings.

**Theorem 2.7** Let \( R \) be a noetherian graded ring, \( q > 0 \) be an integer. \( R \) is a \(^*G_q\)-ring if and only if for all graded prime ideal \( \wp \) of \( R \) such that \( \text{depth} R_{\wp} < q \), \( R_{\wp} \) is a Gorenstein ring.

**Proof:** \( \Rightarrow \) Let \( R \) be a \(^*G_q\)-ring, then:
1) for all \( \wp \in \text{Spec}(R) \), \( \text{depth} R_{\wp} \geq \min\{q, \text{dim} R_{\wp}\} \);
2) for all \( \wp \in \text{Spec}(R) \) such that \( \text{dim} R_{\wp} < q \), \( R_{\wp} \) is a Gorenstein ring.

If there exists \( \wp \in R \) such that \( \text{depth} R_{\wp} < q \) and \( R_{\wp} \) is not a Gorenstein ring then \( \text{dim} R_{\wp} \geq q \) by 2) and by 1) it follows \( \text{depth} R_{\wp} \geq q \). We obtain a contradiction, hence the thesis.

\( \Leftarrow \) We suppose that \( R \) does not satisfy the \((^*S_q)\) condition, that is there exists \( \wp \in \text{Spec}(R) \) such that \( \text{depth} R_{\wp} < \min\{q, \text{dim} R_{\wp}\} \).

- If \( \text{dim} R_{\wp} \leq q \): \( \text{depth} R_{\wp} < \text{dim} R_{\wp} \leq q \). Hence \( R_{\wp} \) is not a Gorenstein ring (because \( \text{depth} R_{\wp} < \text{dim} R_{\wp} \)). We obtain a contradiction because by hypothesis if \( \text{depth} R_{\wp} < q \), then \( R_{\wp} \) is Gorenstein.

- If \( \text{dim} R_{\wp} > q \), then \( \text{depth} R_{\wp} < q \). \( R_{\wp} \) is not a Gorenstein ring (\( \text{depth} R_{\wp} \neq \text{dim} R_{\wp} \)), that is a contradiction by hypothesis. Hence \( R \) satisfies \((^*S_q)\).

Moreover let \( \wp \in \text{Spec}(R) \) such that \( \text{dim} R_{\wp} < q \). Then \( \text{depth} R_{\wp} \leq \text{dim} R_{\wp} < q \). Because \( \text{depth} R_{\wp} < q \), then by hypothesis \( R_{\wp} \) is a Gorenstein ring. Hence \( R \) is a \(^*G_q\)-ring.

Now we introduce the graduate version of the \( q \)-Gorenstein and \( q \)-Bass-Ischisawa rings ([5], [12]).

**Definition 2.8** Let \( R \) be a \(^*S_q\)-ring. \( R \) is said \(^q\)-Gorenstein if \( R_{\wp} \) is a local Gorenstein ring for every \( \wp \in \text{Spec}(R) \) such that \( \text{ht}(\wp) \leq q - 1 \).

**Definition 2.9** Let \( R \) be a \(^*S_q\)-ring. \( R \) is said \(^q\)-Bass-Ischisawa if \( R_{\wp} \) is a local Gorenstein ring for every \( \wp \in \text{Spec}(R) \) such that \( \text{ht}(\wp) = q - 1 \).
Lemma 2.10 Let \((R, m)\) be a local \(\ast S_q\)-ring. Then for all \(\wp \in \text{Spec}(R)\) with \(\text{ht}(\wp) \leq q - 2\), there exists a prime ideal \(Q\) such that \(Q^* \supset \wp^*\) and \(\text{ht}(Q^*) = \text{ht}(\wp^*) + 1\).

Proof: Let \(\wp \in \text{Spec}(R)\) with \(\text{ht}(\wp^*) = \alpha \leq q - 2\). Then \(\wp\) contains a homogeneous regular sequence \(\{x_1, \ldots, x_\alpha\}\) of length \(\alpha = \text{ht}(\wp^*)\). Moreover there exist homogeneous elements \(h_1, \ldots, h_{q-\alpha} \in m\) such that \(\{x_1, \ldots, x_\alpha, h_1, \ldots, h_{q-\alpha}\}\) is a homogeneous regular sequence of length \(q\), because if \(R\) is a \(\ast S_q\)-ring then \(\text{depth} R > q\). In particular, for each \(i \in \{1, \ldots, q - \alpha\}\), \(\{x_1, \ldots, x_\alpha, h_i\}\) is a homogeneous regular sequence of length \(\alpha + 1\). Consider the graded ideal \(J_i = (\wp^*, h_i)\). Then, \(\text{depth}(R/J_i) = \alpha + 1\); in fact \(\{x_1, \ldots, x_\alpha, h_i\}\) is a homogeneous regular sequence of length \(\alpha + 1\) contained in \(J_i\) and it is maximal because for each element \(a \in J_i\), \(a = p + y b_i\) where \(p \in \wp^*\) and \(y \in R\), one has \((x_1, \ldots, x_\alpha, h_i) : a \neq (x_1, \ldots, x_\alpha, h_i)\). Let \(Q \in \text{Spec}(R)\) such that \(Q^* \in \text{Ass}(R/J_i)\) and \(\text{depth}(R/Q^*) = \text{depth}(R/J_i) = \alpha + 1\). Then \(\text{depth}(R/Q^*) \leq q - 1\) because \(\alpha \leq q - 1\). Since \(R\) is a \(\ast S_q\)-ring, it follows \(\text{depth}(R/Q^*) = \text{ht}(Q^*)\). Hence \(\text{ht}(Q^*) = \alpha + 1\) and \(Q^* \supset \wp^*\).

The previous result is extended to non local \(\ast S_q\)-rings.

Lemma 2.11 Let \(R\) be a \(\ast S_q\)-ring all of whose maximal ideals have height \(\geq q\). Then for all \(\wp \in \text{Spec}(R)\) such that \(\text{ht}(\wp^*) \leq q - 2\), there exists a prime ideal \(Q\) such that \(Q^* \supset \wp^*\) and \(\text{ht}(Q^*) = \text{ht}(\wp^*) + 1\).

Proof: By hypothesis on maximal ideals it follows that every \(\wp \in \text{Spec}(R)\) with \(\text{ht}(\wp^*) < q - 1\) is contained at least in one maximal graded ideal \(m\). Then, localizing at \(m\), we obtain a similar situation of Lemma 2.10. Hence the thesis.

Using the previous results, we can state the following:

Theorem 2.12 A local \(\ast q\)-Bass-Ischikawa \((R, m)\) is a \(\ast q\)-Gorenstein ring.

Proof: By hypothesis \(R\) is a \(\ast S_q\)-ring such that \(R_{\wp^*}\) is a Gorenstein ring for every \(\wp \in \text{Spec}(R)\) such that \(\text{ht}(\wp^*) = q - 1\). Then by Lemma 2.10 it follows that for every \(\wp \in \text{Spec}(R)\) such that \(\text{ht}(\wp^*) < q - 1\), there exists an ideal \(Q \in \text{Spec}(R)\) such that \(\text{ht}(Q^*) = q - 1\) and \(Q^* \supset \wp^*\). This means that for every \(\wp \in \text{Spec}(R)\) such \(\text{ht}(\wp^*) < q - 1\), \(R_{\wp^*}\) is a Gorenstein ring. Hence \(R\) is \(\ast q\)-Gorenstein.

This local result can be extended to the non local graduate case.

Corollary 2.13 Let \(R\) be a \(\ast q\)-Bass-Ischikawa ring, all of whose maximal ideals have height \(\geq q\). Then \(R\) is a \(\ast q\)-Gorenstein ring.

Proof: The thesis follows by Theorem 2.12 and Lemma 2.11.
3 Main results

In this section we study some properties of condition (\(*G_q\)) in the passage from \(R/(x)\) to \(R\), where \(R\) is a graded ring and \(x\) is a regular homogeneous element belonging to the homogeneous radical of \(R\).

**Definition 3.1** Let \(R\) be a commutative noetherian graded ring. A homogeneous ideal \(m\) of \(R\) is called *maximal if \(m\) is maximal among the homogeneous ideals of \(R\).

**Example 3.2** Let \(R = K[X_1, \ldots, X_n]\), \(m = (X_1, \ldots, X_n)\) is the only *maximal ideal of \(R\).

**Definition 3.3** Let \(R\) be a commutative noetherian graded ring. The intersection of all *maximal ideals of \(R\) is called homogeneous radical of \(R\).

**Proposition 3.4** Let \(R\) be a commutative noetherian graded ring, \(x\) be a regular homogeneous element belonging to the homogeneous radical of \(R\) and \(S = R/(x)\). If \(S\) satisfies (*\(S_q\)) then \(R\) satisfies (*\(S_q\)).

**Proof:** We prove that for all prime ideal \(\varphi\) of \(R\), depth\(R_{\varphi^*}\) \(\geq\) \(\min\{q, \dim R_{\varphi^*}\}\). Let \(Q\) be a graded prime ideal of \(R\). We have two cases.

I) If \(x \in Q\), \(Q = Q/(x)\) is a prime ideal of \(S\). So we have depth\(R_Q = \dim S_Q + 1\). By hypothesis \(S\) satisfies (*\(S_q\)) that is depth\(S_Q \geq \min\{q, \dim S_Q\}\). It follows:

\[
\text{depth} R_Q = \text{depth} S_Q + 1 \geq \min\{q, \dim S_Q\} + 1 \geq \min\{q, \dim R_Q - 1\} + 1 \geq \min\{q, \dim R_Q\} \quad \text{for all graded prime ideal} \ Q \ \text{of} \ R. \quad \text{Hence the thesis follows because, for every prime ideal} \ \varphi \subset R, \ \varphi^* \ \text{is a graded ideal.}
\]

II) If \(x \notin Q\), \((Q, x) \subset R\) is a graded proper ideal. Let \(N\) be a minimal prime graded ideal over \((Q, x)\) and \(\overline{N} = N/(x)\). Because \(x\) belongs to the homogeneous radical of \(R\), by [3] (Lemma 2.1) we have: depth\((QR_Q) \geq\) depth\(R_N - 1\), then it follows depth\(R_Q \geq\) depth\((QR_Q) \geq\) depth\(R_N - 1 = \dim S_{\overline{N}}\), and by hypothesis \(S\) satisfies (*\(S_q\)), so we have depth\(S_{\overline{N}} \geq\) \(\min\{q, \dim S_{\overline{N}}\} \geq\) \(\min\{q, \dim R_Q\}\), because \(\dim S_{\overline{N}} = \dim R_N - 1\) and \(N \supset (Q, x)\). Hence depth\(R_Q \geq\) \(\min\{q, \dim R_Q\}\), for all graded prime ideal \(Q\) and in particular for \(\varphi^*\). The thesis follows.

**Proposition 3.5** Let \(R\) be a commutative noetherian graded ring and \(x\) be a regular homogeneous element belonging to homogeneous radical of \(R\). Let \(S = R/(x)\) be a (*\(S_{q+1}\))-ring. If \(S\) is a *\(G_q\)-ring, then \(R\) is a *\(G_q\)-ring.

**Proof:** By Proposition 3.4 it follows that \(R\) satisfies (*\(S_{q+1}\)) and so also (*\(S_q\)). It is enough to prove that for all prime ideal \(\varphi\) of \(R\) such that \(\dim R_{\varphi^*} < q\), \(R_{\varphi^*}\) is a Gorenstein ring.
Let $Q$ be a graded prime ideal of $R$. We have two cases.

I) Let $x \in Q$ and $\dim R_Q \leq q + 1$. Let $\overline{Q} = Q/(x)$ be a prime ideal of $S$, so we have $\dim S_{\overline{Q}} \leq q$. Hence by hypothesis it follows that $S_{\overline{Q}}$ is Gorenstein. Then by [1] (Cor. 2.6) we have that $R_Q$ is Gorenstein for all $Q$ graded prime ideal such that $\dim R_Q < q$ and in particular for $\varphi^*$. Then the thesis follows.

II) Let $x \notin Q$, $(Q, x) \subset R$ is a proper graded ideal and $\dim R_Q \leq q$. Let $N$ be a minimal graded prime ideal over $(Q, x)$ and $\overline{N} = N/(x)$. We have $\dim S_{\overline{N}} \leq \min\{q + 1, \dim S_{\overline{Q}}\}$, because $S$ satisfies $(^*S_{q+1})$. If $\dim S_{\overline{N}} \geq q + 1$, because $x$ belongs to the homogeneous radical of $R$, by [3] (Lemma 2.1) we obtain: $q + 1 \leq \dim S_{\overline{N}} = \dim R_N - 1 \leq \dim (QP R_N) \leq \dim (QP R_Q) \leq \text{ht}(Q) = \dim R_Q \leq q$, it is absurd. Then $\dim S_{\overline{N}} < q + 1$ and $\dim R_N \leq q + 1$. Hence as in the case I) $R_N$ is Gorenstein. But $Q \supset N$, so $R_Q$ is Gorenstein too ([1]). We obtain that $R_Q$ is Gorenstein for all graded prime ideal $Q$ such that $\dim R_Q \leq q$, in particular for $\varphi^*$. Hence $R$ is a $^*G_q$-ring.

The converse of the Propositions 3.4 and 3.5 is not true as the following example shows.

**Example 3.6** $R = K[X_1, X_2, X_3]/(X_1 X_2, X_2 X_3, X_1 X_3) = K[x_1, x_2, x_3]$ and $x_1 x_2 = x_2 x_3 = x_1 x_3 = 0$. $R$ is a graded Cohen-Macaulay ring and $\dim R = \text{depth} R = 1$. $m = (x_1, x_2, x_3)$ is the only maximal and prime ideal of $R$ and $\{x_1 + x_2 + x_3\}$ is the maximal regular sequence of length $n = 1$ in $m$. Then $\{x_1 + x_2 + x_3\}$ is a $^*G$-sequence such that the condition $n = 1 \geq \min\{0, \dim R_{\varphi^*}\} = 0$ is true for all prime ideal $\varphi$ of $R$. Hence $R$ is a $G_0$-ring.

Let $S = R/(x_1 + x_2 + x_3)$, where $x_1 + x_2 + x_3$ is a regular homogeneous sequence. $S$ is not $G_0$-ring ([10]).

**Proposition 3.7** Let $R$ be a commutative noetherian graded ring and $x$ be a regular homogeneous element of $R$. If $R$ is a $^*G_q$-ring, for $q > 0$, then $S = R/(x)$ satisfies $(^*G_{q-1})$.

**Proof:** By the hypothesis it follows that $S$ is a $(^*S_{q-1})$-ring. It is enough to show that for every $Q \in \text{Spec}(S)$ such that $\text{ht}(Q^*) \leq q - 1$, $S_{Q^*}$ is Gorenstein. For all $Q \in \text{Spec}(S)$ there exists $\varphi \in \text{Spec}(R)$ such that $S_{Q^*} = R_{\varphi^*}/x R_{\varphi^*}$ and $\text{ht}(\varphi^*) = \text{ht}(Q^*) + 1$. Then, if $\text{ht}(Q^*) \leq q - 1$, $S_{Q^*}$ is Gorenstein since $R_{\varphi^*}$ is so ([6], Corollary 6.6).

**ACKNOWLEDGEMENTS.** We like to thank Professor G. Restuccia of University of Messina and Professor C. Ionescu of University of Bucarest for useful discussions about the topic of this paper.
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Received: May, 2009