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$\mathcal{X}$-Gorenstein Projective Modules

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Abstract

In this paper, we generalize the notion of Gorenstein projective modules. Namely, we introduce $\mathcal{X}$-Gorenstein projective modules, where $\mathcal{X}$ is a class of modules that contains all projective modules. We show that the principal results on Gorenstein projective module remain true for the $\mathcal{X}$-Gorenstein projective modules.

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1 Introduction

Throughout this paper, R denotes a non-trivial associative ring with identity, and all modules are left R-modules.

In 1967-69, Auslander and Bridger [1, 2] introduced the G-dimension for finitely generated R-modules when R is Noetherian, denoted by $\mathrm{G}-\dim(M)$ where M is a finitely generated R-module. As the classical case, the G-dimension of modules is defined in terms of resolutions by modules of G-dimension 0, which are defined as follows:
A finitely generated R-module M has G-dimension 0, if:

- $\mathrm{Ext}^m_R(M,R)=0=\mathrm{Ext}^m_R(\mathrm{Hom}_R(M,R),R)$ for every $m>0$; and

- M is reflexive, that is, the canonical map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is an isomorphism.

In [1], Auslander proved that a finitely generated R-module M has G-dimension 0 if and only if there exists an exact sequence of finitely generated free R-modules $\mathbf{L} = \cdots \to L_1 \to L_0 \to L^0 \to L^1 \to \cdots$ such that $M \cong \mathrm{Im}(L_0 \to L^0)$ and the complex $\mathrm{Hom}_R(\mathbf{L}, R)$ is exact.
In [5, 6], Enochs and Jenda defined, over arbitrary rings, the Gorenstein projective modules as follows:

Definition 1.1 An R-module M is said to be Gorenstein projective, if there exists an exact sequence of projective R-modules

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective R-module.
The exact sequence $\mathbf{P}$ is called a *complete projective* resolution.

Over a Noetherian ring R, Avramov, Buchweitz, Martsinkovsky, and Reiten proved that a finitely generated R-module M is Gorenstein projective if and only if $\mathrm{G}-\dim(M) = 0$ (see [4, Theorem 4.2.6] and [4, notes p. 99]). So, the notion of Gorenstein projective modules is an extension of the notion of modules of G-dimension 0. Furthermore, the Gorenstein projective modules share many nice properties of the classical projective module (see, for instance, [4, 8, 7]). In this paper, we show that some of these results remain true whenever we consider, in Definition 1.1, Q to be in any class of modules containing all projective modules. Namely, we define $\mathcal{X}$-Gorenstein projective modules, where $\mathcal{X}$ is a class of R-modules that contains all projective R-modules (see Definition 2.1). In Proposition 2.2 we characterize the $\mathcal{X}$-Gorenstein projective modules. Our main result is Theorem 2.3, in which, we study the behavior of the notion of $\mathcal{X}$-Gorenstein projective modules in short exact sequences. We end the paper with a characterization of rings over which every R-module is $\mathcal{X}$-Gorenstein projective. These rings are particular cases of the well-known quasi-Frobenius rings.

2 $\mathcal{X}$-Gorenstein projective modules

In this paper we investegate the following generalization of Gorenstein projective modules.

Definition 2.1 *Let $\mathcal{X}$ be a class of R-modules that contains all projective R-modules. An R-module M is called $\mathcal{X}$-Gorenstein projective, if there exists*

an exact sequence of projective R-modules $\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ *such that* $M \cong \mathrm{Im}(P_0 \to P^0)$ *and* $\mathrm{Hom}_R(\mathbf{P}, F)$ *is exact whenever* $F \in \mathcal{X}$.
The sequence $\mathbf{P}$ *is called an* $\mathcal{X}$*-complete projective resolution.*

We start with the following characterization of an $\mathcal{X}$-Gorenstein projective module.

Proposition 2.2 *For an R-module M, the following conditions are equivalent:*

1. *M is* $\mathcal{X}$*-Gorenstein projective.*
2. **i)** $\mathrm{Ext}^i_R(M, F) = 0$ *for every* $F \in \mathcal{X}$ *and every* $i > 0$;

   ii) *There exists an exact sequence of R-modules* $\mathbf{Q} = 0 \to M \to P_0 \to P_1 \to \cdots$, *where each* P_i *is projective, such that* $\mathrm{Hom}_R(\mathbf{Q}, F)$ *is exact for every* $F \in \mathcal{X}$.
3. *There exists a short exact sequence of R-modules* $0 \to M \to P \to N \to 0$, *where P is projective and N is* $\mathcal{X}$*-Gorenstein projective.*
4. *There exists a family of short exact sequences of R-modules* $0 \to M_i \to P_i \to M_{i+1} \to 0$ $(i \in \mathbb{Z})$, *where each* P_i *is projective and* $M_0 = M$, *such that* $\mathrm{Ext}^1_R(M_i, F) = 0$ *for every* $F \in \mathcal{X}$ *and every* $i \in \mathbb{Z}$.

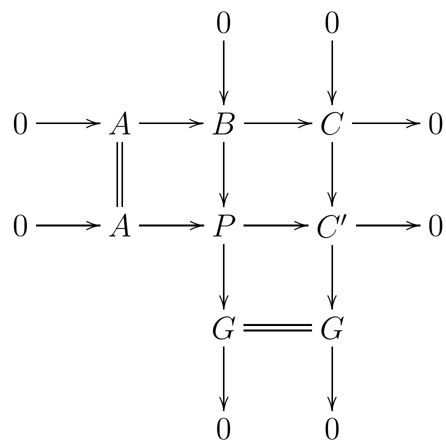
Proof. The proof of the equivalences (1) ⇔ (2) ⇔ (4) is analogous to the ones of the Gorenstein projective counterpart (see [4, 8]).
The implication (3) ⇒ (4) is obvious.
To end, we prove the implication (3) ⇒ (2). Let $F \in \mathcal{X}$. Applying the functor $\mathrm{Hom}_R(-, F)$ to the exact sequence $0 \to M \to P \to N \to 0$, we get the long exact sequence: $\cdots \to \mathrm{Ext}^i_R(N, F) \to \mathrm{Ext}^i_R(P, F) \to \mathrm{Ext}^i_R(M, F) \to \cdots$ For every $i > 0$, we have: $\mathrm{Ext}^i_R(N, F) = 0$ (since N is $\mathcal{X}$-Gorenstein projective and by the equivalent (1) ⇔ (2)). Also, we have $\mathrm{Ext}^i_R(P, F) = 0$ (since P is projective). Then, $\mathrm{Ext}^i_R(M, F) = 0$ for every $i > 0$.
It remains to prove (*ii*). Since N is $\mathcal{X}$-Gorenstein projective and by the equivalent (1) ⇔ (2), there exists an exact sequence of R-modules $\mathbf{P} = 0 \to N \to P_0 \to P_1 \to \cdots$, where each P_i is projective, such that $\mathrm{Hom}_R(\mathbf{P}, F)$ is exact for all R-modules $F \in \mathcal{X}$. Assembling this sequence with the short exact sequence $0 \to M \to P \to N \to 0$ we get the following exact sequence $\mathbf{Q} = 0 \to M \to P \to P_0 \to P_1 \to \cdots$ such that the sequence $\mathrm{Hom}_R(\mathbf{Q}, F)$ is exact for every R-module $F \in \mathcal{X}$, as desired. ■

The following result, which investigates the behavior of $\mathcal{X}$-Gorenstein projective modules in short exact sequences, generalizes [8, Theorem 2.5].

Theorem 2.3 *1. Let $0 \to A \to B \to C \to 0$ be a short exact sequences of R-modules, where C is $\mathcal{X}$-Gorenstein projective. Then, A is $\mathcal{X}$-Gorenstein projective if and only if B is $\mathcal{X}$-Gorenstein projective.*

2. Let $(M_i)_{i\in I}$ be a family of R-modules. Then, $\bigoplus_{i\in I} M_i$ is $\mathcal{X}$-Gorenstein projective if and only if M_i is $\mathcal{X}$-Gorenstein projective for every $i \in I$.

Proof. The equivalences of both (1) and (2) can be proved similarly to the one of [8, Theorem 2.5]. Here, we give a new and simple proof of the "only if" part of (1). Then, assume that B is $\mathcal{X}$-Gorenstein projective. By Proposition 2.2 (1) $\Leftrightarrow$ (3), there exists an exact sequence of R-modules $0 \to B \to P \to G \to 0$, where P is projective and G is $\mathcal{X}$-Gorenstein projective. Consider the following pushout diagram:



Applying the "if" part to the right vertical short exact sequence, we get that C' is $\mathcal{X}$-Gorenstein projective. Therefore, use the equivalence (1) $\Leftrightarrow$ (3) of Proposition 2.2 and the middle horizontal short exact sequence to get that A is $\mathcal{X}$-Gorenstein projective. ■

We end the paper with a characterization of rings over which every R-module is $\mathcal{X}$-Gorenstein projective. These rings are particular cases of the well-known quasi-Frobenius rings.

Proposition 2.4 *Every R-module is $\mathcal{X}$-Gorenstein projective if and only if every R-module in $\mathcal{X}$ is injective.*
In particular, if the above equivalence conditions are satisfied, then R is quasi-Frobenius.

Proof. First, from [3, Theorem 2.2] and its proof, if one of the equivalence conditions are satisfied, then R is quasi-Frobenius.

Now, assume that every R-module is $\mathcal{X}$-Gorenstein projective. Then, from Proposition 2.2, $\mathrm{Ext}^i_R(M,F)=0$ for every R-module M, every $F\in\mathcal{X}$, and every $i>0$. Then, every F in $\mathcal{X}$ is injective.
Conversely, consider an R-module M. Let $\cdots\to P_1\to P_0\to M\to 0$ and $0\to M\to I_0\to I_1\to\cdots$ be projective and injective resolutions of M. Since, by the reason above, R is quasi-Frobenius, every injective R-module is projective. Then, the above injective resolution is a right projective resolution of M. Now, assembling the two above resolutions, we get the following exact sequence: $\cdots\to P_1\to P_0\to I_0\to I_1\to\cdots$ Since, by hypothesis, every R-module in $\mathcal{X}$ is injective, the above exact sequence is clearly an $\mathcal{X}$-complete projective resolution, as desired. ■

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