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π-Generalized Semi-Preclosed Sets

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Abstract. In this paper, we introduce a new class of sets called π-generalized semi-preclosed, briefly πgsp-closed. We study some of its basic properties. This class of sets is strictly placed between the class of πgp-closed sets and the class of $gspr$-closed sets.

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1. Introduction and preliminaries

Throughout this paper (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a (X, τ) space, $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ and A^c denote the closure of A, the interior of A and the complement of A in X, respectively.

We recall the following definitions, which are useful in the sequel.

Definition 1.1. *A subset A of a space X is called:*
(i) semi-open [5] *if* $A \subset \mathrm{Cl}(\mathrm{Int}(A))$.
(ii) preopen [6] *if* $A \subset \mathrm{Int}(\mathrm{Cl}(A))$.
(iii) α-open [7] *if* $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$.

(iv) β-open [1] *(=semi-preopen* [2]*) if* $A \subset \mathrm{Cl}\,(\mathrm{Int}\,(\mathrm{Cl}\,(A)))$.
(v) regular open [9] *if* $A = \mathrm{Int}\,(\mathrm{Cl}\,(A))$.
(vi) π-open [10] *if A is the union of regular open sets.*

The complements of the above mentioned sets are called semi-closed, preclosed, α-closed, β-closed (= semi-preclosed), regular closed and π-closed, respectively.

The preclosure (resp. semi-preclosure) of a subset A of X, denoted by $p\,\mathrm{Cl}(A)$ (resp. $sp\,\mathrm{Cl}(A)$) is the intersection of all preclosed (resp. semi-preclosed) sets containing A. The semi-preinterior of A, denoted by $sp\,\mathrm{Int}(A)$ is the union of all semi-preopen sets contained in A.

It is well known that semi-open sets as well as preopen sets are semi-preopen, any union of semi-preopen sets is semi-preopen, and that the intersection of a semi-preopen set with an α-open set is semi-preopen.

Definition 1.2. *A subset A of a space X is called:*
(i) πg-closed [3] *if* $\mathrm{Cl}\,(A) \subset U$ *whenever* $A \subset U$ *and U is π-open.*
(ii) πgp-closed [8] *if* $p\,\mathrm{Cl}\,(A) \subset U$ *whenever* $A \subset U$ *and U is π-open.*
(iii) $gspr$-closed [4] *if* $sp\,\mathrm{Cl}\,(A) \subset U$ *whenever* $A \subset U$ *and U is regular open.*

The complement of a πg-closed (resp. πgp-closed, $gspr$-closed) set is said to be πg-open (resp. πgp-open, $gspr$-open).

2. π-Generalized semi-preclosed sets

Definition 2.1. *A subset A of a space X is called π-generalized semi-preclosed (briefly πgsp-closed) if* $sp\,\mathrm{Cl}\,(A) \subset U$ *whenever* $A \subset U$ *and U is π-open. The complement of a πgsp-closed set is called πgsp-open. The π-kernel (π-$\ker(A)$) of A is the intersection of all π-open sets containing A.*

Remark 2.2. *A subset A of a space (X, τ) is πgsp-closed if and only if* $sp\,\mathrm{Cl}(A) \subset \pi\text{-}\ker(A)$.

Remark 2.3. *Every semi-preclosed set is πgsp-closed.*

Remark 2.4. *Every πgsp-closed is $gspr$-closed.*

Proposition 2.5. *Every πgp-closed set is πgsp-closed.*

Proof. Let A be πgp-closed in (X, τ) and $A \subset U$ where U is π-open. Since A is πgp-closed, $p\,\mathrm{Cl}(A) \subset U$. Since every preclosed set is semi-preclosed, $sp\,\mathrm{Cl}(A) \subset p\,\mathrm{Cl}(A)$. Therefore, $sp\,\mathrm{Cl}(A) \subset U$. Hence, A is πgp-closed. □

The following diagram summarizes the implications among the introduced concept and other related concepts.

$$\begin{array}{ccccc} \pi g\text{-closed} & \rightarrow & \pi gp\text{-closed} & & \\ & & \downarrow & & \\ \text{semi-preclosed} & \rightarrow & \pi gsp\text{-closed} & \rightarrow & gspr\text{-closed} \end{array}$$

Remark 2.6. *The following three examples show that the converses of Remarks 2.3, 2.4 and Proposition 2.5 are not true in general.*

Example 2.7. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, \{a\}, X\}$. *Then the set* $\{a\}$ *is* πgsp*-closed in* (X, τ) *(observe that the only regular open (π-open) set containing* $\{a\}$ *is* X*). However,* $\{a\}$ *is not semi-preclosed.*

Example 2.8. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. *Then the set* $\{a, b\}$ *is gspr-closed in* (X, τ) *(observe that the only regular open set containing* $\{a, b\}$ *is* X*). However,* $\{a, b\}$ *is not* πgsp*-closed (observe that* $\{a\}, \{b\}$ *are regular open and thus* $\{a, b\}$ *is* π*-open, also* $\{a, b\} \subset \{a, b\}$ *and* $sp\,\mathrm{Cl}(\{a, b\}) = X \not\subset \{a, b\}$*).*

Example 2.9. *Let* $X = \{a, b, c, d\}$ *and* $\tau = \{\phi, \{a, b\}, \{c\}, \{a, b, c\}, X\}$. *Then the set* $\{a, b\}$ *is* πgsp*-closed in* (X, τ) *as it is semi-preclosed. However,* $\{a, b\}$ *is not* πgp*-closed (observe that* $\{a, b\}$ *is regular open and thus* π*-open, also* $\{a, b\} \subset \{a, b\}$ *and* $p\,\mathrm{Cl}(\{a, b\}) = \{a, b, d\} \not\subset \{a, b\}$*).*

3. Some Properties of πgsp-closed sets

Proposition 3.1. *If* A *is* π*-open and* πgsp*-closed, then* A *is semi-preclosed.*

Proof. Since A is π-open and πgsp-closed, $sp\,\mathrm{Cl}(A) \subset A$, but $A \subset sp\,\mathrm{Cl}(A)$, so $A = sp\,\mathrm{Cl}(A)$. Hence, A is semi-preclosed. □

Proposition 3.2. *Let* A *be a* πgsp*-closed in* (X, τ). *Then* $sp\,\mathrm{Cl}(A)\backslash A$ *does not contain any nonempty* π*-closed set.*

Proof. Let F be a nonempty π-closed subset of $sp\,\mathrm{Cl}(A)\backslash A$. Then $A \subset X\backslash F$, where A is πgsp-closed and $X\backslash F$ is π-open. Thus $sp\,\mathrm{Cl}(A) \subset X\backslash F$, or equivalently, $F \subset X\backslash sp\,\mathrm{Cl}(A)$. Since by assumption $F \subset sp\,\mathrm{Cl}(A)$, we get a contradiction. □

Corollary 3.3. *Let* A *be* πgsp*-closed in* (X, τ). *Then* A *is semi-pre-closed if and only if* $sp\,\mathrm{Cl}(A)\backslash A$ *is* π*-closed.*

Proof. **Necessity.** Let A be πgsp-closed. By hypothesis $sp\,\mathrm{Cl}(A) = A$ and so $sp\,\mathrm{Cl}(A)\backslash A = \phi$ which is π-closed.
Sufficiency. Suppose $sp\,\mathrm{Cl}(A)\backslash A$ is π-closed. Then by Proposition 3.2, $sp\,\mathrm{Cl}(A)\backslash A = \phi$, that is, $sp\,\mathrm{Cl}(A) = A$. Hence, A is semi-preclosed. □

Proposition 3.4. *If* A *is a* πgsp*-closed subset of* (X, τ) *such that* $A \subset B \subset sp\,\mathrm{Cl}(A)$, *then* B *is also* πgsp*-closed subset of* (X, τ).

Proof. Let U be a π-open set in (X, τ) such that $B \subset U$. Then $A \subset U$. Since A is πgsp-closed, then $sp\,\mathrm{Cl}(A) \subset U$. Now, since $sp\,\mathrm{Cl}(A)$ is semi-preclosed, $sp\,\mathrm{Cl}(B) \subset sp\,\mathrm{Cl}(sp\,\mathrm{Cl}(A)) = sp\,\mathrm{Cl}(A) \subset U$. Therefore, B is also a πgsp-closed. □

Proposition 3.5. *For every point x of a space X, $X \backslash \{x\}$ is πgsp-closed or π-open.*

Proof. Suppose $X \backslash \{x\}$ is not π-open. Then X is the only π-open set containing $X \backslash \{x\}$. Hence, $X \backslash \{x\}$ is πgsp-closed. □

Definition 3.6. *A space X is called πgsp if every πgsp-closed subset of X is semi-preclosed.*

Proposition 3.7. *A space X is πgsp if and only if every singleton of X is π-closed or semi-preopen.*

Proof. **Necessity.** Let $x \in X$ be such that $\{x\}$ is not π-closed. Then $X \backslash \{x\}$ is not π-open, and thus $X \backslash \{x\}$ is πgsp-closed. By assumption, $X \backslash \{x\}$ is semi-preclosed, that is, $\{x\}$ is semi-preopen.

Sufficiency. Let A be a πgsp-closed subset of X and let $x \in sp\,\mathrm{Cl}(A)$. By assumption, we have the following two cases:

(i) x is semi-preopen, but $x \in sp\,\mathrm{Cl}(A)$, so $x \in A$.

(ii) x is π-closed. Then by Proposition 3.2, $x \notin sp\,\mathrm{Cl}(A) \backslash A$, but $x \in sp\,\mathrm{Cl}(A)$, so $x \in A$.

Hence, $sp\,\mathrm{Cl}(A) = A$, that is, A is semi-preclosed. □

4. πgsp-OPEN SETS

The following corollary is an immediate consequence of the fact that $sp\,\mathrm{Cl}(X \backslash A) = X \backslash sp\,\mathrm{Int}(A)$.

Corollary 4.1. *A set A in a topological space (X, τ) is πgsp-open if and only if $F \subset sp\,\mathrm{Int}(A)$ whenever F is π-closed in (X, τ) and $F \subset A$.*

Corollary 4.2. *If $sp\,\mathrm{Int}(A) \subset B \subset A$ and A is πgsp-open, then B is πgsp-open.*

Proof. Follows from Corollary 4.1. □

Proposition 4.3. *If a set A is πgsp-open in a topological space (X, τ), then $G = X$ whenever G is π-open in (X, τ) and $sp\,\mathrm{Int}(A) \cup A^c \subset G$.*

Proof. Suppose that G is π-open and $sp\,\mathrm{Int}(A) \cup A^c \subset G$. Now $G^c \subset sp\,\mathrm{Cl}(A^c) \cap A = sp\,\mathrm{Cl}(A^c) \backslash A^c$. Since G^c is π-closed and A^c is πgsp-closed, by Proposition 3.2, $G^c = \phi$ and hence $G = X$. □

Proposition 4.4. *Let A be πgsp-open in (X,τ) and let B be α-open. Then $A\cap B$ is πgsp-open in (X,τ).*

Proof. Let F be any π-closed subset of X such that $F\subset A\cap B$. Hence $F\subset A$ and by Corollary 4.1, $F\subset sp\,\mathrm{Int}(A)=\bigcup\{U : U$ is semi-preopen and $U\subset A\}$. Obviously, $F\subset\bigcup(U\cap B)$, where U is a semi-preopen set in X contained in A. Since $U\cap B$ is a semi-preopen set contained in $A\cap B$ for each semi-preopen set U contained in A, $F\subset sp\,\mathrm{Int}(A\cap B)$, and by Corollary 4.1, $A\cap B$ is πgsp-open in X. □

Proposition 4.5. *Let (X,τ) be a topological space and $A,B\subset X$. If B is πgsp-open and $sp\,\mathrm{Int}(B)\subset A$, then $A\cap B$ is πgsp-open.*

Proof. Since B is πgsp-open and $sp\,\mathrm{Int}(B)\subset A$, $sp\,\mathrm{Int}(B)\subset A\cap B\subset B$. By Corollary 4.2, $A\cap B$ is πgsp-open. □

Proposition 4.6. *Let (X,τ) be a space such that the family $SPO(X,\tau)$ of all semi-preopen subsets of (X,τ) be closed under finite intersections. If A and B are πgsp-open in (X,τ), then $A\cap B$ is πgsp-open.*

Proof. Let $X\backslash(A\cap B)=(X\backslash A)\cup(X\backslash B)\subset U$, where U is π-open. Then $X\backslash A\subset U$ and $X\backslash B\subset U$. Since A and B are πgsp-open, $sp\,\mathrm{Cl}(X\backslash A)\subset U$ and $sp\,\mathrm{Cl}(X\backslash B)\subset U$. By hypothesis, $sp\,\mathrm{Cl}((X\backslash A)\cup(X\backslash B))=sp\,\mathrm{Cl}(X\backslash A)\cup sp\,\mathrm{Cl}(X\backslash B)\subset U$. Hence, $A\cap B$ is πgsp-open. □

Proposition 4.7. *If $A\subset X$ is πgsp-closed, then $sp\,\mathrm{Cl}(A)\backslash A$ is πgsp-open.*

Proof. Let A be πgsp-closed and let F be a π-closed set such that $F\subset sp\,\mathrm{Cl}(A)\backslash A$. Then by Proposition 3.2, $F=\phi$. So, $F\subset sp\,\mathrm{Int}(sp\,\mathrm{Cl}(A)\backslash A)$. By Corollary 4.1, $sp\,\mathrm{Cl}(A)\backslash A$ is πgsp-open. □

The following Lemma can be easily verified.

Lemma 4.8. *For every subset A of a space (X,τ), $sp\,\mathrm{Int}(sp\,\mathrm{Cl}(A)\backslash A)=\phi$.*

Proposition 4.9. *Let $A\subset B\subset X$ and let $sp\,\mathrm{Cl}(A)\backslash A$ be πgsp-open. Then $sp\,\mathrm{Cl}(A)\backslash B$ is also πgsp-open.*

Proof. Suppose $sp\,\mathrm{Cl}(A)\backslash A$ is πgsp-open and let F be a π-closed subset of (X,τ) with $F\subset sp\,\mathrm{Cl}(A)\backslash B$. Then $F\subset sp\,\mathrm{Cl}(A)\backslash A$. By Corollary 4.1 and Lemma 4.8, $F\subset sp\,\mathrm{Int}(sp\,\mathrm{Cl}(A)\backslash A)=\phi$. Thus, $F=\phi$ and hence, $F\subset sp\,\mathrm{Int}(sp\,\mathrm{Cl}(A)\backslash B)$. □

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