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A Simple Differential Equation System for the Description of Competition among Religions

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Abstract

A set of three differential equation (DE) is presented which describes the increase and decrease in the number of followers of two competing religions R_1 and R_2 and a irreligious group R_3. The DE is formed similar to usual models for competing populations and for prey-predator systems. Linear birth and death rates, linear gain and loss terms and quadratic interaction terms are included. The model is extremely simple but nevertheless only numerical solutions can be given. No description of actual socio-religious developments is indented but the formulation of a starting point for better models.

Mathematics Subject Classification: 91C99

Keywords: population dynamics, religions, conversion, adherence, differential equation system

1 The differential equation system

Consider two religions R_1 and R_2. Their numbers of followers are $x_1(t)$ and $x_2(t)$ where t denotes the (continuous) time. We want to describe the development of $x_1(t)$ and $x_2(t)$ with time t. For that purpose, we set up a differential equation system DE which will connect the functions $x_1(t)$ and $x_2(t)$ with their time derivatives $dx_1(t)/dt$ and $dx_1(t)/dt$. The following assumptions enter into DE.

a) R_1 and R_2 enjoy birth rates α_1 and α_2 per time unit and suffer from the same birth rate ω. Thus, we will play with different birth rates which will mimic the influence of fertility on $dx_1(t)/dt$ and $dx_1(t)/dt$.

b) R_1 and R_2 interact and attract or detract followers from each other. Thus, we are confronted with processes of conversions from R_1 to R_2 and vice versa and we speak of interreligious processes. As usual in population dynamics [1] we represent a mutual interaction as a product of the population numbers $x_1(t)$ and $x_2(t)$. However, we introduce coefficients γ_{ij} and γ_{ji} which describe the strength of the mutual interaction between R_1 and R_2. Hereby γ_{12} is the gain of R_1 from conversion of followers from R_1 to R_2 and γ_{21} is the corresponding loss of R_1 to R_2. Such products $x_i x_j$ are typical for fox-rabit models in population dynamics and for chemical reactions (law of mass action).

c) R_1 and R_2 can gain and loose followers in non-interreligios processes. These non-interreligious processes are described as proportional to the population numbers $x_1(t)$ and $x_2(t)$ and may be called intrareligious processes. The proportional coefficients are called ϵ_{ij} and ϵ_{ji} and they describe the strength of the intrareligious processes. Hereby ϵ_{ij} means gain for religion i from religion j and ϵ_{ji} the converse. The interreligious process within R_i $(i, j = 1, 2, 3)$ depends on the number of followers $x_i(t)$ and thus the algebraic expression is $\epsilon_{ij}x_j$ for gains of R_i from R_j and $\epsilon_{ji}x_i$ for losses of R_i to R_j. For example, R_1 can gain members of R_2 according to $\epsilon_{12}x(2)$, that is proportional to the number of followers of R_2. Conversely, R_1 can loose members to R_2 according to $\epsilon_{12}x(2)$, that is proportional to its own number of followers. To our best knowledge, these "conversion terms" $\epsilon_{ij}x_j$ form a new ansatz in population dynamics.

d) Each religion R_i has a core of members xa_i. These followers are adherent to their religion, i.e. they do not change their religion.

e) A third formal religion comes into play called R_3. Its members do not have a religious believe, we will call them irreligious.

Then the following differential equation system DE results

$$
\begin{aligned}
\frac{dx_1(t)}{dt} &= \alpha_1 x_1(t) + \gamma_{12}(xa_1 + x_1(t))x_2(t) - \gamma_{21}x_1(t)(xa_2 + x_2(t)) \\
&\quad + \epsilon_{12}x_2(t) - \epsilon_{21}x_1(t) + \gamma_{13}(xa_1 + x_1(t))x_3(t) - \gamma_{31}x_1(t)(xa_3 + x_3(t)) \\
&\quad + \epsilon_{13}x_3(t) - \epsilon_{31}x_1(t) - \omega(xa_1 + x_1(t)) \qquad (1) \\
\frac{dx_2(t)}{dt} &= \alpha_2 x_2(t) + \gamma_{21}x_1(t)(xa_2 + x_2(t)) - \gamma_{12}(xa_1 + x_1(t))x_2(t) \\
&\quad + \epsilon_{21}x_1(t) - \epsilon_{12}x_2(t) + \gamma_{23}(xa_2 + x_2(t))x_3(t) - \gamma_{32}x_2(t)(xa_3 + x_3(t)) \\
&\quad + \epsilon_{23}x_3(t) - \epsilon_{32}x_2(t) - \omega(xa_2 + x_2(t)) \qquad (2) \\
\frac{dx_3(t)}{dt} &= \alpha_3 x_3(t) + \gamma_{31}x_1(t)(xa_3 + x_3(t)) - \gamma_{13}(xa_1 + x_1(t))x_3(t) \\
&\quad + \epsilon_{31}x_1(t) - \epsilon_{13}x_3(t) + \gamma_{32}x_2(t)(xa_3 + x_3(t)) - \gamma_{23}(xa_2 + x_2(t))x_3(t) \\
&\quad + \epsilon_{32}x_2(t) - \epsilon_{23}x_3(t) - \omega(xa_3 + x_3(t)). \qquad (3)
\end{aligned}
$$

An analytical solution of the DE is not possible, thus we can study numerical solutions only. These numerical solutions depend on the values chosen for the coefficients. In this study we do not aim to model actual conversion rates between actual religions. Such a modeling would pose difficulties not only on the model itself (i.e. on the DE) but also would be challenging what concerns the empirical data.

Instead we want to present some numerical solutions for reasonable values of the coefficients. Furthermore, the solutions depend on the starting conditions $x_1(t=0)$, $x_2(t=0)$, and $x_3(t=0)$ at $t=0$. In fact we have a rather particular situation in mind: We will assume that $x_1(t=0) > x_2(t=0)$, that is, R_1 is the "bigger" (maybe also "older") religion and R_2 the smaller (maybe also "younger") religion at $t=0$. In addition we will assume that in general the followers of R_1 are likely to convert either to R_2 or to R_3. On the other hand, the followers of R_2 are in general more adherent to R_2 and thus less likely for conversion to R_1 or R_3. Finally, R_3 is understood as a sink for followers from R_1 or R_2. Thus, we are faced with a situation in which R_1 is decreasing in numbers of followers and R_2 is increasing. Then the main questions in the view of R_1 is: For which set of coefficients $\mathcal{SC} = \alpha_1, \ldots, \omega$ and at which final time t_f will R_1 vanish, i.e. $x_1(t=t_f=0)$? For which values of $\mathcal{SC}$ is an equilibrium possible among R_1 and R_2, i.e. $x_1(t) = x_2(t)$? Is there any hope for R_1?

A realistic selection of values of the coefficients and of the starting utilizes empirical data on conversion rates. Such data are available [2, 3]. However, we extract only rough order-of-magnitude estimates from the literature and we do not perform any further analysis of these values. A detailed discussion on available conversion rates can be found in [4].

2 Numerical solutions

The examination of the numerical solutions of DE can be divided into several cases, we will discuss two of them. In the first case we will suppose that interreligious interactions do not take place, thus all $\gamma_{ij} = \gamma_{ji} = 0$. This assumption leads to a much simpler, linear differential equation system LDE. The second case is formed by our general DE including the interreligious interaction, i.e. the quadratic terms $x_i(t)x_j(t)$. As will be seen, the behavior of the DE strongly depends on the quadratic terms. Thus, these terms need a particular justification when used for realistic models or even for forecasts.

The birth rates α_i are understood as fractions of the populations per year with a descendant. Our main focus is that R_2 is much more fertile than R_1. For R_3 we put a rather low fertility, of cause this setting is arguable. Just for simplicity we put $\omega_1 = \omega_2 = \omega_3 = \omega$ which seems reasonable, we do not expect different mortality rates per year for R_1, R_2, and R_3. The most difficult to estimate parameters are the $\gamma_{i,j}$ and $\epsilon_{i,j}$. In fact, a great part of the further development of the DE would consist in a more detailed and realistic theory of the coefficients $\gamma_{i,j}$ and $\epsilon_{i,j}$. A very rough order-of-magnitude estimation for $\gamma_{i,j}$ and $\epsilon_{i,j}$ can be taken from published conversion rates. However, for the present study we use more or less arbitrary values for $\gamma_{i,j}$ and $\epsilon_{i,j}$. The starting conditions $x_i(t=0)$ are chosen in order to reflect a situation in which R_1 has the majority of followers and R_2 is still a small community of followers. In addition, there is a certain fraction of irreligious people R_3, thus in total $x_1(t=0) > x_3(t=0) > x_2(t=0)$. Finally, we have to meet assumptions on the cores xa_i of believers who do not change their religions. Again, we furnish R_1 with weaker conditions than R_2, that is followers of R_2 are more adherent to their religion. For example, we put $xa_1 = 0.5x_1(t=0)$ and $xa_2 = 0.95x_2(t=0)$. However, experimenting with the xa_i exhibits a small influence on the general behavior of DE and thus we also use $xa_i = 0$.

2.1 The linear case

A first inspection of empirical conversion rates among religions shows low conversion rates. Thus, the interreligious interaction may be small, at least in peaceful times. Therefore, in the first step we set $\gamma_{ij} = \gamma_{ji} = 0$. On the other hand, presently some religions suffer from a considerable drain of followers to the non-religious R_3. Then again, some other religions do not suffer from any conversion to R_3. We will set up a situation in which R_1 has remarkable losses to R_3 and R_2 has none. Most interestingly is the balance of gain and loss among R_1 and R_2. We will assume more conversions from R_1 to R_2 then vice versa, thus $\epsilon_{12} < \epsilon_{21}$.

Then from (1) - (3) the linear differential equation system

$$\frac{dx_1(t)}{dt} = \alpha_1 x_1(t) + \epsilon_{12}x_2(t) - \epsilon_{21}x_1(t) + \epsilon_{13}x_3(t) - \epsilon_{31}x_1(t) - \omega(xa_1 + x_1(t)) \tag{4}$$

$$\frac{dx_2(t)}{dt} = \alpha_2 x_2(t) + \epsilon_{21}x_1(t) - \epsilon_{12}x_2(t) + \epsilon_{23}x_3(t) - \epsilon_{32}x_2(t) - \omega(xa_2 + x_2(t)) \tag{5}$$

$$\frac{dx_3(t)}{dt} = \alpha_3 x_3(t) + \epsilon_{31}x_1(t) - \epsilon_{13}x_3(t) + \epsilon_{32}x_2(t) - \epsilon_{23}x_3(t) - \omega(xa_3 + x_3(t)) \tag{6}$$

results. If we use the following starting values for the variables $x_1(t=0) = 750$, $x_2(t=0) = 50$, $x_3(t=0) = 300$ together with $xa_1 = 0.5x_1(t=0)$, $xa_2 = 0.95x_2(t=0)$, $xa_3 = 0.0x_3(t=0)$ and if we use the following concrete values for the coefficients $\alpha_1 = 0.01$, $\alpha_2 = 0.02$, $\alpha_3 = 0.003$ and $\epsilon_{1,2} = 0.0$, $\epsilon_{1,3} = 0.001$, $\epsilon_{2,3} = 0.01$, $\epsilon_{2,1} = 0.01$, $\epsilon_{3,1} = 0.1$, $\epsilon_{3,2} = 0.0$ and finally $\omega = 0.005$, then we get from (4) - (6)

$$\frac{dx_1(t)}{dt} = -0.105x_1(t) - 1.875 + 0.001x_3(t) \tag{7}$$

$$\frac{dx_2(t)}{dt} = 0.015x_2(t) + 0.01x_1(t) - 0.2375 + 0.01x_3(t) \quad (8)$$

$$\frac{dx_3(t)}{dt} = -0.013x_3(t) + 0.1x_1(t). \quad (9)$$

A numerical solution for the time period $t = 0, \ldots, 100$ years is displayed in figure 1. The continuous line belongs to R_1, the broken line belongs to R_2, and the doted line belongs to R_3.

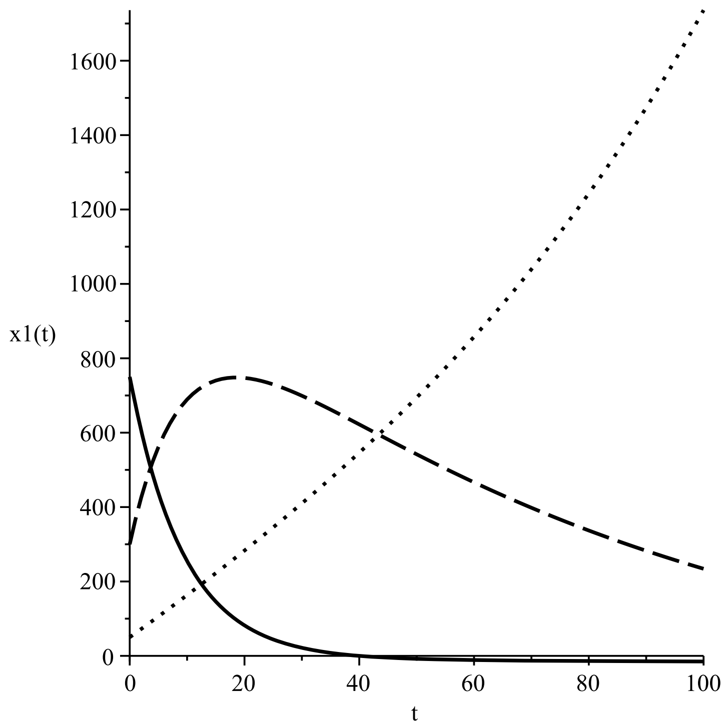

Figure 1: Solution to (7) - (9) .

It should be stressed again that no forecast is intended. We are interested in the behavior of the DE. For that purpose figure 1 describes an extreme case, that is the decease of R_1. Obviously, if there is a constant flow of followers from R_1 to R_2 and R_3 and no sufficient gain then the decease is unavoidable. The decrease for R_1 is dramatically, the increase for R_2 and R_3 is considerable. However, after a certain period of a majority for R_3, this religion will also decline.

It is possible to solve (4) - (6) for conditions under which R_1, R_2 and R_3 are in equilibrium, that is $dx_1(t)/dt = dx_2(t)/dt = dx_3(t)/dt = 0$. As there are three equations we can solve (4) - (6) for a set of three of its coefficients. The values of the remaining coefficients have to be set according to the situation in mind. For example, we could find a solution for $\alpha_1(t), \epsilon_{12}(t), \epsilon_{13}(t)$ and obviously such a solution will depend on time t. Numerical experience shows that the resulting values for $\alpha_1(t), \epsilon_{12}(t), \epsilon_{13}(t)$ can become rather unrealistic. For example, at a late point in time the system (7) - (9) can get into equilibrium only for a very high fertility rate α_1.

However, our main interest is not the equilibrium but to reach a stable state for R_1. For that purpose we allow for changes in the conditions over time. A possible development could be an equalization of the exchange coefficients ϵ_{ij} over time, that is $\epsilon_{ij}(t = \inf) \asymp \epsilon(ji)(t = \inf)$. As a first formulation we use

$$\epsilon_{ij} = (1 - \exp(-4t))\epsilon_{ji} \quad (10)$$

were the damping factor $\exp(-4t))$ is chosen rather arbitrarily again (together with $x_1(t = 0) = 750$, $x_2(t = 0) = 50$, $x_3(t = 0) = 300$, and with $xa_1 = 0.5x_1(t = 0)$, $xa_2 = 0.95x_2(t = 0)$, $xa_3 = 0.0x_3(t = 0)$, and with $\alpha_1 = 0.01$, $\alpha_2 = 0.02$, $\alpha_3 = 0.003$ and with $\epsilon_{1,2} = 0.0$, $\epsilon_{1,3} = 0.001$, $\epsilon_{2,3} = 0.01$, $\epsilon_{2,1} = 0.01$, $\epsilon_{3,1} = 0.1$, $\epsilon_{3,2} = 0.0$, and finally with $\omega = 0.005$).

The result is depicted in figure 2.

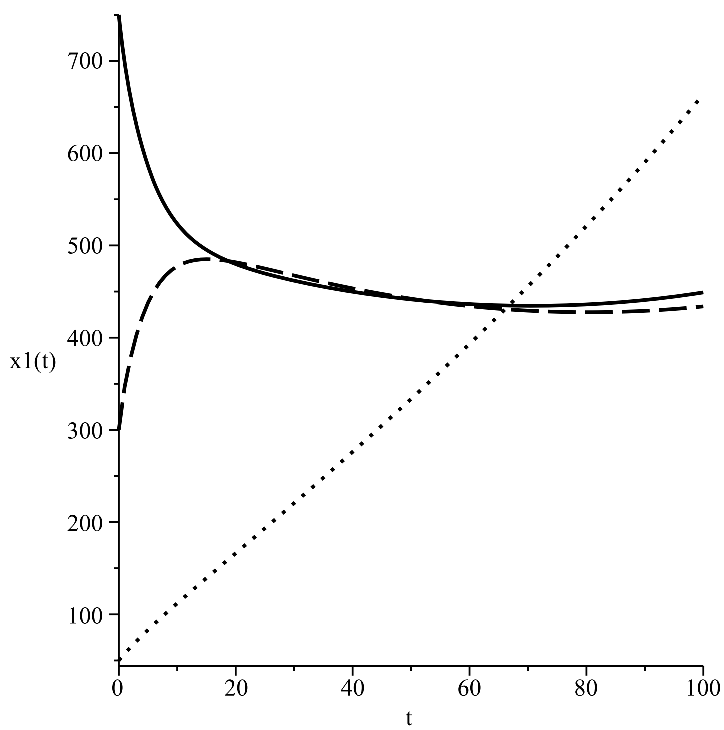

Figure 2: Solution to (7) - (9) with additional damping factors (10).

For the linear system (4) - (6) it is even possible to find closed-form solutions for $x_1(t)$, $x_2(t)$, $x_3(t)$. We will net repeat them here, in principle they are superpositions of exponential functions. This is not a surprise, as the solution of $dx(t)/dt = x(t)$ is the exponential function $x(t) = \exp(t)$.

2.2 The quadratical case

If we allow for interaction coefficients $\gamma_{ij} \neq 0$, then we arrive at the full DE from (1) - (3). The effect of the interaction terms $\gamma_{ij}x_i x_j$ on the solution behavior of (1) - (3) is strong compared to the exchange terms $\epsilon_{ij}x_j$ since their interaction terms are of quadratical nature. For the next example, the γ-coefficients were $\gamma_{12} = 0.00001$, $\gamma_{13} = 0.0001$, $\gamma_{21} = 0.0001$, $\gamma_{31} = 0.0002$, $\gamma_{32} = 0.0$. The only damping factors used in that example were

$$\alpha_2 = \alpha_1 + 0.02\exp(-t) \tag{11}$$

$$\gamma_{12} = (1 - \exp(-t))\gamma_{21}. \tag{12}$$

Then from (1) - (3) follows (with $x_1(t=0) = 750$, $x_2(t=0) = 50$, $x_3(t=0) = 300$ and with $xa_1 = 0.5x_1(t=0)$, $xa_2 = 0.95x_2(t=0)$, $xa_3 = 0.0x_3(t=0)$ and with $\alpha_1 = 0.01$, $\alpha_2 = 0.02$, $\alpha_3 = 0.005$ and with $\epsilon_{1,2} = 0.0$, $\epsilon_{1,3} = 0.001$, $\epsilon_{2,3} = 0.01$, $\epsilon_{2,1} = 0.01$, $\epsilon_{3,1} = 0.1$, $\epsilon_{3,2} = 0.0$ and finally with $\omega = 0.005$.)

$$\begin{aligned}\frac{dx_1(t)}{dt} &= -0.105x_1(t) + (0.0001 - 0.0001\exp(-t))(375.0 + x_1(t))x_2(t) \\ &\quad -0.0001x_1(t)(50 + x_2(t)) - 1.8750 + (0.0001(375.0 + x_1(t)))x_3(t) \\ &\quad -0.0002x_1(t)x_3(t) + 0.001x_3(t) \qquad (13)\\ \frac{dx_2(t)}{dt} &= (0.01 + 0.02\exp(-t))x_2(t) + 0.0001x_1(t)(50 + x_2(t)) \\ &\quad -(0.0001 - 0.0001\exp(-t))(375.0 + x_1(t))x_2(t) + 0.01x_1(t) \\ &\quad -0.250 + (0.0001(50 + x_2(t)))x_3(t) + 0.01x_3(t) - 0.005x_2(t) \qquad (14)\\ \frac{dx_3(t)}{dt} &= -0.011x_3(t) + 0.0002x_1(t)x_3(t) - (0.0001(375.0 + x_1(t)))x_3(t) \\ &\quad +0.1x_1(t) - (0.0001(50 + x_2(t)))x_3(t) \qquad (15)\end{aligned}$$

As can be seen from figure 3 the mutual interactions lead to dramatical decreases or increases for x_1, x_2, x_3 in the times span of the first (roughly) 15, 30 and 45 years. Interestingly, the population number x_1 can recover after a certain time span and stabilizes at a new, but lower level. This recovery is due to the damping factors (11) - (12). Under the same conditions but without equalization x_1 vanishes.

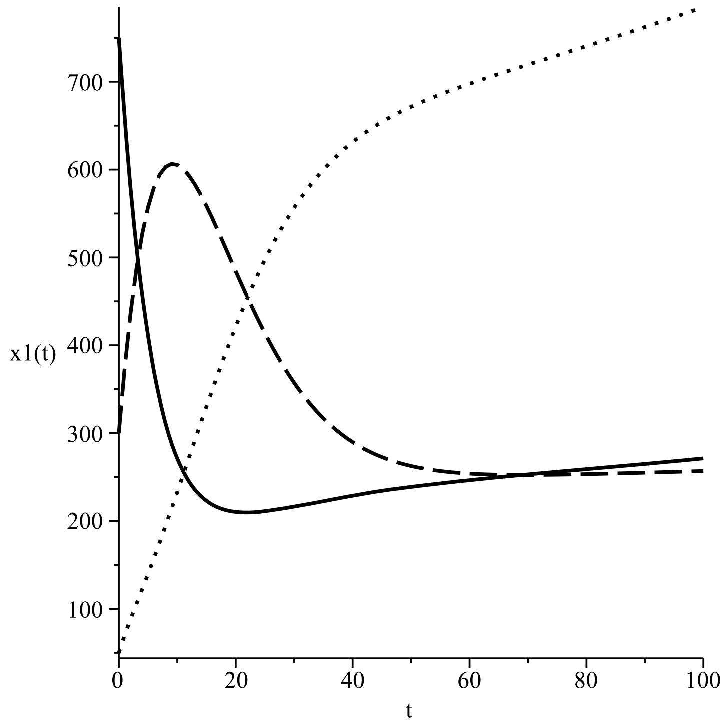

Figure 3: Recovery of x_1 due to (11) - (12).

3 Discussion and Conclusion

Our system DE is formulated similar to models in biological population dynamics. In fact, without the exchange terms $\epsilon_{ij}x_j$ the system has the same structure as the well-known fox-rabbit model. However, we have several new features build into (1) - (3). The main of them is the use of exchange terms. In our context, these exchanges correspond to conversions among religions where the gain for R_i from R_j is proportional to R_j and the loss of R_i to R_j is proportional to R_i. This ansatz allows the modeling of intrareligious processes like the dissolution of an religion which leads to a flow of their followers to other religions or to the group of irreligious persons. A total dissolution may be prevented by the presence of a faithful core of followers. Such a feature is included in the DE. Another new feature is the inclusion of time-dependent coefficients. For the beginning we just used damping factors which cause an equalization between coefficients like birth rates. If our DE should gain an empirical meaning for the description of religious populations, then the challenge is to find more realistic expressions for the coefficients. A further particular feature of (1) - (3) is the incorporation of a third party, that is the religion R_3 which designates the irreligious persons which do not share any believe. The circumstance of more than two parties leads to the effect that for any mutual coefficient ζ_{ij} we may have $\zeta_{ij} \neq \zeta_{ji}$.

At the present stage, our system (1) - (3) is just a proposition for further research. As the first step one will have to find more realistic values for the coefficients which may prove to be difficult. A further, fairly challenging step would be the inclusion of a spatial dependency. Some insight into the dynamics of religious populations can be found at the present stage already. The religion R_1 (which plays the role of the declining religion) can get into some stable state and survive the interaction with the increasing religion R_2. Within the model, the equalization of the fertilities coefficients and of the interaction coefficients with time provide several routes to get into a stable state. We believe that the system (1) - (3) comprises many possible developments among the population numbers of interacting religions.

Hayward ([5, 6]) developed a model for church growth basing on differential equations similar to equations used in epidemics. Other methods than differential equation may be used to model competing religions. One may think of agent-based methods, probabilistic descriptions based on a master equation, Monte Carlo simulations of bit-string etc. For example, the opinion dynamics and the study of competing languages, two closely related fields of socio-physics, use several such methods [7]. The DE has the advantage of conceptional simpleness.

4 Appendix

The following commands will set up and solve the DE (1) - (3) within the computer algebra system Maple (version 13). Concrete values for the starting conditions $x_i(t = 0)$ and for the coefficients α_i, ϵ_{ij}, γ_{ij}, ω with $i, j = 1, 2, 3$ and $i \neq j$ must be given in advance which is not displayed here.

```
restart; with(plots); with(DEtools);

DE1 := diff(x1(t), t) = alpha1*x1(t)+gamma12*(x1a+x1(t))*x2(t)
-gamma21*x1(t)*(x2a+x2(t))+epsilon12*x2(t)-epsilon21*x1(t)
+gamma13*(x1a+x1(t))*x3(t)-gamma31*x1(t)*(x3a+x3(t))
+epsilon13*x3(t)-epsilon31*x1(t)-omega*(x1a+x1(t));

DE2 := diff(x2(t), t) = alpha2*x2(t)+gamma21*x1(t)*(x2a+x2(t))
-gamma12*(x1a+x1(t))*x2(t)+epsilon21*x1(t)-epsilon12*x2(t)
+gamma23*(x2a+x2(t))*x3(t)-gamma32*x2(t)*(x3a+x3(t))
+epsilon23*x3(t)-epsilon32*x2(t)-omega*(x2a+x2(t));

DE3 := diff(x3(t), t) = alpha3*x3(t)+gamma31*x1(t)*(x3a+x3(t))
```

```
-gamma13*(x1a+x1(t))*x3(t)+epsilon31*x1(t)-epsilon13*x3(t)
+gamma32*x2(t)*(x3a+x3(t))-gamma23*(x2a+x2(t))*x3(t)
+epsilon32*x2(t)-epsilon23*x3(t)-omega*(x3a+x3(t));

ABPsys:= [DE1, DE2, DE3];

aplot:=DEplot(ABPsys, [x1(t),x2(t),x3(t)], t = 0..100,
[[x1(0)=x1t0,x2(0)=x2t0,x3(0)=x3t0]], scene=[t,x1(t)],
thickness=2, linestyle=1, linecolor=black, stepsize=1.0):
bplot:=DEplot(ABPsys, [x1(t),x2(t),x3(t)], t = 0..100,
[[x1(0)=x1t0,x2(0)=x2t0,x3(0)=x3t0]], scene=[t,x2(t)],
thickness=2, linestyle=2, linecolor=black, stepsize=1.0):
pplot:=DEplot(ABPsys, [x1(t),x2(t),x3(t)], t = 0..100,
[[x1(0)=x1t0,x2(0)=x2t0,x3(0)=x3t0]], scene=[t,x3(t)],
thickness=2, linestyle=3, linecolor=black, stepsize=1.0):
display([aplot,bplot,pplot]);
```

The following Maple commands will solve the LDE (4) - (6) in analytical form. Concrete values for the coefficients must be given in advance which is not displayed here.

```
ics := x1(0) = 750, x2(0) = 50, x3(0) = 300;
dsolve(‘union‘({DE1, DE2, DE3}, {ics}), {x1(t), x2(t), x3(t)});
```

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