Generalized Numerical Radii Inequalities for Hilbert Space Operators

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Abstract
In this paper, we establish several generalized numerical radii inequalities for Hilbert space operators using some classical inequalities for nonnegative real numbers and some operator inequalities.

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1 Introduction
Let $H$ be a complex Hilbert space with inner product $\langle ., . \rangle$, and $B(H)$ the set of all bounded linear operators on $H$. We note that $B(H)$ is an involutive algebra where the involution $A \to A^*$ on $B(H)$ is simply the map of $A$ onto its Hilbert space adjoint. For $A \in B(H)$, let $w(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of $A$, respectively. The numerical radius is defined by

$$w(A) = \sup \{ \|\lambda\| : \lambda \in W(A) \},$$

where $W(A)$ is the numerical range of $A$ given by $W(A) = \{ \langle Ax, x \rangle : x \in H, \|x\| = 1 \}$. The usual operator norm of $A$, $\|A\|$ is defined by

$$\|A\| = \sup_{\|x\|=1} \|Ax\|, \text{ for all } x \in H,$$

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where $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$. It is well known that $w(.)$ defines a norm on $B(H)$ and that for every $A \in B(H)$, we have

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|.$$  \hfill (1)

Thus, the usual operator norm and the numerical radius norm are equivalent. The inequalities in (1) are sharp: if $A^2 = 0$, then the first inequality becomes an equality, while the second inequality becomes an equality if $A$ is normal. In fact, for a nilpotent operator $A$ with $A^n = 0$, Haagerup and Harpe [12] show that $w(A) \leq \|A\| \cos\left(\frac{\pi}{n+1}\right)$. In particular, when $n = 2$, we get the reverse inequality of the first inequality in (1). For a comprehensive account of theory of the numerical range and numerical radius, we refer the reader to [5, 8, 9].

The study of the numerical radii inequalities has been of great interest to mathematicians in recent times and for the most recent results on this topic, see [3], [4], [6] or [7].

The Schwarz inequality for positive operators asserts that if $A$ is a positive operator in $B(H)$, then

$$|\langle Ax, y \rangle|^2 \leq \langle Ax, x \rangle \langle Ay, y \rangle,$$

for all $x, y \in H$. \hfill (2)

For an arbitrary $A$ in $B(H)$, a ‘mixed Schwarz’ inequality has been established in [11]. This inequality asserts that

$$|\langle Ax, y \rangle|^2 \leq \langle |A|^{2\alpha}x, x \rangle \langle |A^*|^{2(1-\alpha)}y, y \rangle$$

for all $x, y$ in $H$ and for $0 \leq \alpha \leq 1$. Here $|A| = (A^*A)^{\frac{1}{2}}$ and $|A^*| = (AA^*)^{\frac{1}{2}}$.

An important consequence of (3) is the widely celebrated Heinz inequality [1, 11] which says that if $T, A$ and $B$ are operators in $B(H)$ such that $A$ and $B$ are positive and $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for all $x, y \in H$, then

$$|\langle Tx, y \rangle| \leq \|A^{\alpha}x\| \|B^{1-\alpha}y\|$$

for $0 \leq \alpha \leq 1$. \hfill (4)

The main ingredients in the proof of (3) given in [8, 11] are the polar decomposition and elementary aspects of spectral theorem. A generalized version of (3) has been obtained in [2] using the idea of positivity and the self-adjointness of certain operator matrices defined on $H \oplus H$. In the next section of this paper, we prove several generalized numerical radii inequalities for operators on a Hilbert space using some classical inequalities for nonnegative real numbers and some operator inequalities.

## 2 Numerical radius Inequalities

In order to state our results, we shall need the following well-known Lemmas. The first Lemma follows from spectral Theorem for positive operators and Jensen’s inequality [2].
Lemma 2.1. Let $A \in B(H)$ be positive operator and let $x \in H$ be any unit vector. Then
\[
\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle \text{ for all } r \geq 1, \text{ and }
\]
\[
\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r \text{ for all } 0 < r \leq 1.
\]

The second Lemma also follows from spectral Theorem and is due to Kittaneh [2].

Lemma 2.2. For a self-adjoint operator $A$ in $B(H)$,
\[
|\langle Ax, x \rangle| \leq \langle |A|x, x \rangle \text{ for all } x \in H.
\]

The third Lemma is the famous Clarkson’s inequality.

Lemma 2.3. If $a, b \in \mathbb{C}$, then
\[
2(|a|^r + |b|^r) \leq |a + b|^r + |a - b|^r, \text{ for } r \geq 2.
\]

The fourth Lemma is concerned with positive real numbers and is a simple consequence of the classical Jensen’s inequality concerning the convexity of the function $f(t) = t^r$, $r \geq 1$.

Lemma 2.4. If $a$ and $b$ are nonnegative real numbers, then
\[
(a + b)^r \leq 2^{r-1}(a^r + b^r), \text{ for } r \geq 1.
\]

Our first result is the following generalization,

Theorem 2.5. Let $A$ and $B$ be self-adjoint operators in $B(H)$, and $r \geq 1$. Then
\[
w^r(A + B) \leq 2^{r-1}\||A|^r + |B|^r\|.
\]

Proof. For every unit vector $x \in H$, we have
\[
|\langle (A + B)x, x \rangle|^r = |\langle Ax, x \rangle + \langle Bx, x \rangle|^r
\leq (|\langle Ax, x \rangle| + |\langle Bx, x \rangle|)^r
\leq (\langle |A|x, x \rangle + \langle |B|x, x \rangle)^r
\leq 2^{r-1}(\langle |A|x, x \rangle^r + \langle |B|x, x \rangle^r)
\leq 2^{r-1}(\langle |A|^r x, x \rangle + \langle |B|^r x, x \rangle)
= 2^{r-1}(\langle (|A|^r + |B|^r)x, x \rangle).
\]

The result therefore follows by taking supremum over all unit vectors in $H$. \qed
Corollary 2.6. Letting \( r = 1 \) in (5), we obtain

\[ w(A + B) \leq ||A| + |B||. \]  \hspace{1cm} (6)

Corollary 2.7. If \( A = B \), then the inequality (6) reduces to the second inequality of (1).

Corollary 2.8. Letting \( r = 2 \) in (5), we obtain

\[ w^2(A + B) \leq 2||A|^2 + |B|^2||. \]  \hspace{1cm} (7)

Corollary 2.9. If \( A = B \) in (7), we obtain the second inequality of (1).

In the next result we use Clarkson’s inequality to give another generalization of the numerical radius of the sum of two operators.

Theorem 2.10. Let \( A \) and \( B \) be self-adjoint operators in \( B(H) \), and \( r \geq 2 \). Then

\[ w^r(A + B) \leq 2^{r-2}||A + B|^r + |A - B|^r||. \]  \hspace{1cm} (8)

**Proof.** From Theorem 2.5 above, we have

\[ w^r(A + B) \leq 2^{r-1}||A|^r + |B|^r||. \]  \hspace{1cm} (9)

But from Clarkson’s inequality, see Lemma 2.3, we have

\[ |A|^r + |B|^r \leq \frac{1}{2}(|A + B|^r + |A - B|^r), \text{ for } r \geq 2. \]

So that equation (9) becomes

\[
\begin{align*}
  w^r(A + B) & \leq 2^{r-1}||A|^r + |B|^r|| \\
  & \leq 2^{r-1}||\frac{1}{2}(|A + B|^r + |A - B|^r)|| \\
  & = 2^{r-1}2^{-1}||A + B|^r + |A - B|^r|| \\
  & = 2^{r-2}||A + B|^r + |A - B|^r||.
\end{align*}
\]

\[ \square \]

Remark 2.11. The inequality (5) is sharper than the inequality (8) because

\[ w^r(A + B) \leq 2^{r-1}||A|^r + |B|^r|| \leq 2^{r-2}||A + B|^r + |A - B|^r||. \]

Corollary 2.12. If \( A = B \), then the inequality (8) reduces to

\[ w^r(A) \leq 2^{r-2}||A||^r. \]  \hspace{1cm} (10)
Corollary 2.13. Letting $r = 2$ in (10), we obtain the second part of inequality (1).

Corollary 2.14. Letting $r = 2$ in (8), we obtain

$$w^2(A + B) \leq ||A + B||^2 + ||A - B||^2.$$  

The next result considers the famous Heinz inequality and generalizes the numerical radius inequality for Hilbert space operators.

Theorem 2.15. Let $T, A$ and $B$ be operators in $B(H)$ such that $A$ and $B$ are positive and $\|Tx\| \leq \|Ax\|$, $\|T^*x\| \leq \|Bx\|$ for all $x \in H$, $0 \leq \alpha \leq 1$, $r \geq 1$, then

$$w^r(T) \leq ||AA^*|^{\alpha r}|BB^*|^{(1-\alpha)r}||^{\frac{1}{2}}.$$  

Proof. By the inequality (4), we have

$$|\langle Tx, x \rangle|^{2r} \leq \|A^\alpha x\|^{2r}\|B^{1-\alpha}x\|^{2r}$$

$$= \langle A^\alpha x, A^\alpha x \rangle^{r}\langle B^{1-\alpha}x, B^{1-\alpha}x \rangle^{r}$$

$$= \langle (A^\alpha)^*A^\alpha x, x \rangle^{r}\langle (B^{1-\alpha})^*B^{1-\alpha}x, x \rangle^{r}$$

$$= \langle (A^\alpha)^*A^\alpha x, x \rangle^{r}\langle (B^*)^{1-\alpha}x, x \rangle^{r}$$

$$\leq \langle A^\alpha A^\alpha x, x \rangle^{r}\langle B^*B^{1-\alpha}x, x \rangle^{r}$$

$$\leq \langle A^\alpha A^\alpha x, x \rangle^{r}\|B^*B^{1-\alpha}x, x \rangle$$ by Lemma 2.1,

$$\leq \langle A^\alpha A^\alpha A^\alpha x, x \rangle^{r}\|B^*B^{1-\alpha}x, x \rangle^{r}.$$ 

Now the result follows immediately by taking supremum over all unit vectors in $H$. 

Our last result in this paper is another generalization of the numerical radius following the Heinz inequality even though the inequality (11) is sharper.

Theorem 2.16. Let $T, A$ and $B$ be defined as in Theorem 2.15. Then

$$w^r(T) \leq \frac{1}{2}||AA^*|^{\alpha r} + |BB^*|^{(1-\alpha)r}||.$$  

Proof. By the inequality (4), we have

$$|\langle Tx, x \rangle|^{2r} \leq \|A^\alpha x\|^{2r}\|B^{1-\alpha}x\|^{2r}$$

$$\leq \langle (A^\alpha)^*A^\alpha |x, x \rangle\langle (B^{1-\alpha})^*B^{1-\alpha}|r, x \rangle$$

$$\leq \frac{1}{4}(\langle (A^\alpha)^*A^\alpha |x, x \rangle + \langle (B^{1-\alpha})^*B^{1-\alpha}|r, x \rangle)^2$$ (by the arithmetic-geometric mean inequality)

$$\leq \frac{1}{4}(\langle A^\alpha A^\alpha x, x \rangle + \langle B^*B^{(1-\alpha)r}x, x \rangle)^2$$

The remaining part of the proof is trivial.
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