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Abstract

In this paper, one introduces an Ishikawa iterative scheme for finding a common element of the set of solution of an equilibrium problem and the set of fixed points of a strong pseudo-contraction mapping in a real Hilbert space. Some strong convergence theorems are established using the scheme. Furthermore, the error estimate of the scheme is given.

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1 Introduction and Preliminary

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $K$ be a nonempty closed convex subset of $H$. A mapping $T$ of $K$ into itself is called a $k$-strong pseudo-contraction mapping, if there exists a constant $k \in (0, 1)$ such that $\forall \ x, y \in K$, $\langle Tx - Ty, x - y \rangle \leq (1 - k)\|x - y\|^2$. By literatures [1,2], a strong pseudo-contraction mapping has unique fixed point in $K$. Let $F(T)$ denote the set of fixed points of $T$(i.e. $F(T) = \{ x \in K : Tx = x \}$). A mapping $S$ of $K$ into itself is called a $L$-Lipschitz mapping, if there exists a constant $L > 1$ such that $\forall \ x, y \in K$, $\|Sx - Sy\| \leq L\|x - y\|$. Let $f$ be a bifunction of $K \times K$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real number. The equilibrium problem for $f : K \times K \to \mathbb{R}$ is to find $x \in K$ such that

$$f(x, y) \geq 0, \ \forall \ y \in K.$$ (1)
We use \( EP(f) \) to denote the set of solution of the problem (1). If \( F(x, y) = \langle Ax, y - x \rangle \), here \( A : K \to K \) is a nonlinear operator, then the problem (1) becomes the following classical variational inequality problem:

Find \( x \in K \) such that

\[
\langle Ax, y - x \rangle \geq 0, \quad \forall \ y \in K. \tag{2}
\]

This shows that problem (2) is a special case of the problem (1).

Because many problems in physics (optimization or economics) may reduce to solve the problem (1), recently, a lot of author introduced some iterative schemes to find a common element of \( EP(f) \) and the fixed point set of an nonlinear operator, such as non-expansive mapping, quasi-non-expansive mappings and strict pseudo-contraction mapping, see [3-10]. These iterative schemes under some conditions have been proved that they have strong convergence, but the authors do not give the error estimate of the iterative schemes. In this paper, we introduce a Ishikawa iterative scheme to find a common element of the solution set of the equilibrium problem (1) and the fixed point set of a strong pseudo-contraction mapping, we prove not only the iterative scheme with strong convergence but also give it’s error estimate.

For the sequence \( \{x_n\} \) in \( H \), we write \( x_n \rightharpoonup x \) to indicate that \( \{x_n\} \) converges weakly to \( x \). \( x_n \to x \) implies that \( \{x_n\} \) converges strongly to \( x \).

For solving the equilibrium problem (1) for a bifunction \( f : K \times K \to \mathbb{R} \), let us assume that \( f \) satisfy the following conditions:

(A1) \( f(x, x) = 0 \) for all \( x \in K \);
(A2) \( f \) is monotone, that is, \( f(x, y) + f(y, x) \leq 0 \) for all \( x, y \in K \);
(A3) for each \( x, y, z \in K \), \( \lim_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x, y) \);
(A4) for each \( x \in K, \ y \mapsto f(x, y) \) is convex and lower semi-continuous.

The following Lemmas will be used in this paper.

**Lemma 1.1** ([11]). Let \( K \) be a nonempty convex subset of \( H \) and \( f \) be a bifunction of \( K \times K \) into \( \mathbb{R} \) satisfying (A1) – (A4). Let \( r > 0 \) and \( x \in H \). Then, there exists \( z \in K \) such that

\[
f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \text{ for all } y \in K.
\]

**Lemma 1.2** ([12]). Assume that \( f \) is a bifunction of \( K \times K \) into \( \mathbb{R} \) satisfying (A1) – (A4). For \( r > 0 \) and \( x \in H \), define a mapping \( T_r : H \to K \) as follows:

\[
T_r(x) = \left\{ z \in K : f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall \ y \in K \right\}
\]

for all \( x \in H \). Then the following hold:

(1) \( T_r \) is single-valued;
(2) $T_r$ is firmly non-expansive, that is, for any $x, y \in H$,
$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ is closed and convex.

**Lemma 1.3** ([13]). Let $\{\alpha_n\} \subset [0, 1]$ and $\{a_n\}$ be a sequence of nonnegative real number satisfying the following relation:
$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \ n \geq 0,$$
if (i) $\sum \alpha_n = \infty$; (ii) $\lim \sup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$, $\sum \gamma_n < \infty$, then $\lim a_n = 0$.

**Lemma 1.4.** Let $H$ be a Hilbert space, then $\|x + y\|^2 \leq \|y\|^2 + 2\langle x, x + y \rangle$, $\forall \ x, y \in H$.

## 2 Main Results

**Theorem 2.1.** Let $K$ be a nonempty closed convex subset of $H$ and $F$ a bifunction from $K \times K$ to $\mathbb{R}$ satisfying (A1) – (A4). $T$ is a $k$-strong pseudo-contraction and $L$-Lipschitz mapping of $K$ into $K$ such that $\Omega = EP(f) \cap F(T) \neq \emptyset$. For given $x_1 \in K$, let $\{x_n\}$ and $\{u_n\}$ be generated by

$$
\begin{align*}
\begin{cases}
\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \forall \ y \in K, \\
\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n Ty_n, \\
\ y_n = (1 - \beta_n)u_n + \beta_n Tu_n,
\end{cases}
\end{align*}
$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1)$ and $r_n$ satisfy the following conditions:

(i) $\alpha_n \leq \min \{k^2, \frac{1}{(2+k)(1-k)}; \frac{k(1-k)}{2L(1+L)}\}$, $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\beta_n \leq \frac{k(1-k)}{2L(1+L)}$; (iii) $r_n > 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $p \in \Omega$, respectively. Moreover, $\{x_n\}$ admits error estimate
$$\|x_n - p\| \leq \frac{1}{k} \sqrt{\exp^{-k^2 \sum_{i=1}^{n} \alpha_i} \|x_1 - Tx_1\|}.$$

**Proof.** Let $p \in \Omega$. Lemma 1.2 shows $u_n = T_{r_n}x_n$ and $\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|$. From Lemma 1.4 and (3), we have

$$
\begin{align*}
\|x_{n+1} - p\|^2 &= \|(1 - \alpha_n)(u_n - p) + \alpha_n(Ty_n - p)\|^2 \\
&\leq (1 - \alpha_n)^2\|u_n - p\|^2 + 2\alpha_n\langle Ty_n - p, x_{n+1} - p \rangle \\
&= (1 - \alpha_n)^2\|u_n - p\|^2 + 2\alpha_n\langle Ty_n - Tx_{n+1} + Tx_{n+1} - p, x_{n+1} - p \rangle \\
&\leq (1 - \alpha_n)^2\|u_n - p\|^2 + 2\alpha_n(1 - k)\|x_{n+1} - p\|^2 + 2\alpha_nL\|y_n - x_{n+1}\|\|x_{n+1} - p\|. \tag{4}
\end{align*}
$$
On the other hand, since $\alpha_n \leq \frac{k(1-k)}{2L(1+L)}$, $\beta_n \leq \frac{k(1-k)}{2L(1+L^2)}$, then

$$
\|x_{n+1} - y_n\| = \|(\beta_n - \alpha_n)(u_n - p) + \alpha_n(Ty_n - p) + \beta_n(p - Tu_n)\|
\leq (\beta_n + \alpha_n)\|u_n - p\| + \beta_n L\|u_n - p\| + \alpha_n L\|y_n - p\|
\leq (\beta_n + \alpha_n)\|u_n - p\| + \beta_n L\|u_n - p\| + \alpha_n L(1 - \beta_n + \beta_n L)\|u_n - p\|
\leq (\beta_n + \alpha_n)(L + 1) + \beta_n L(L + 1)\|u_n - p\|
\leq \frac{k(1-k)}{L}\|u_n - p\|. \tag{5}
$$

Substituting (5) into (4), then

$$
\|x_{n+1} - p\|^2 \leq (1 - \alpha_n)^2\|u_n - p\|^2 + 2\alpha_n(1 - k)\|x_{n+1} - p\|^2 + 2\alpha_n k(1 - k)\|u_n - p\|^2\|x_{n+1} - p\|
\leq (1 - \alpha_n)^2\|u_n - p\|^2 + \alpha_n(2 + k)(1 - k)\|x_{n+1} - p\|^2 + 2\alpha_n k(1 - k)\|u_n - p\|^2\|x_{n+1} - p\|^2
\leq (1 - \alpha_n(2 - k))\|x_n - p\|^2 + \alpha_n(2 + k)(1 - k)\|x_{n+1} - p\|^2.
\tag{6}
$$

Since $\alpha_n \leq \min\{k^2, \frac{1}{(2+k)(1-k)}\}$, then from (6) we have

$$
\|x_{n+1} - p\|^2 \leq \frac{1 - \alpha_n(2 - k)}{1 - \alpha_n(2 + k)(1 - k)}\|x_n - p\|^2
= \left\{1 - \frac{k^2\alpha_n}{1 - \alpha_n(2 + k)(1 - k)}\right\}\|x_n - p\|^2
\leq (1 - k^2\alpha_n)\|x_n - p\|^2
\leq \sqrt{\exp(-k^2\sum_{i=1}^{n}\alpha_i)}\|x_1 - p\|. \tag{7}
$$

From (7) we know that $\{x_n\}$ converge strongly to $p \in \Omega$, so is $\{u_n\}$.

Since $T$ is a $k$–strong pseudo-contraction, hence the next inequality holds:

$$
\|x - y\| \leq \|(x - y) + \lambda[(I - T + kI)x - (I - T + kI)y]\|, \forall x, y \in C, \forall \lambda > 0, \tag{8}
$$

see[1]. If we take $\lambda = \frac{1}{k}$, $x = x_1, y = p$ in (8), then we have $\|x_1 - p\| \leq \frac{1}{k}\|x_1 - Tx_1\|$. Thus, the sequence $\{x_n\}$ admits error estimate

$$
\|x_n - p\| \leq \frac{1}{k}\sqrt{\exp(-k^2\sum_{i=1}^{n}\alpha_i)}\|x_1 - Tx_1\|. \tag{9}
$$

This completes proof of Theorem 2.1.

As a direct consequence of theorem 2.1, the next corollary 2.2 holds:

**Corollary 2.2.** Let $K$ be a nonempty closed convex subset of $H$ and $F$ be a bifunction from $K \times K$ to $\mathbb{R}$ satisfying (A1) – (A4). $T$ is a $k$–strong pseudo-contraction and $L$–Lipschitz mapping of $K$ into $K$ such that $\Omega = EP(f) \cap F(T) \neq \emptyset$. For given $x_1 \in K$, let $\{x_n\}$ and $\{u_n\}$ be generated by

$$
\begin{cases}
  f(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \forall y \in K, \\
x_{n+1} = (1 - \alpha_n)u_n + \alpha_n Tu_n,
\end{cases} \tag{10}
$$
where \( \{\alpha_n\}, \{\beta_n\} \subset [0, 1) \) and \( r_n \) satisfy the following conditions:

(i) \( \alpha_n \leq \min\{k^2, \frac{1}{(2+k)(1-k)}, \frac{k(1-k)}{2L(1+L)}\} \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \); (ii) \( r_n > 0 \).

Then \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( p \in \Omega \), respectively. Moreover, \( \{x_n\} \) admits error estimate

\[
\|x_n - p\| \leq \frac{1}{k} \sqrt{\exp^{-k^2 \sum_{i=1}^{n} \alpha_i}} \|x_1 - Tx_1\|. \tag{11}
\]

Recall that a mapping \( T : K \to K \) is called \( k \)--strong monotone mapping, if there exists \( 0 < k < 1 \) such that \( \langle Tx - Ty, x - y \rangle \geq k\|x - y\|^2 \), \( x, y \in K \). It is very clear that \( T \) is \( k \)--strong monotone mapping if and only if \( I - T \) is \( k \)--strong pseudo-contraction mapping. Hence, we may consult an iterative scheme such that it converge strongly to a common element of the set of solution of an equilibrium problem and the set of fixed points of a strong monotone mapping in a real Hilbert space. More precisely, we obtain the following Theorem.

**Theorem 2.3.** Let \( K \) be a nonempty closed convex subset of \( H \) and \( F \) be a bifunction from \( K \times K \) to \( \mathbb{R} \) satisfying (A1) -- (A4). \( T \) is a \( k \)--strong monotone mapping of \( K \) into \( K \) such that \( I - T \) is \( L \)--Lipschitz mapping and \( \Omega = EP(f) \cap N(T) \neq \emptyset \), where \( N(T) = \{x \in K : Tx = 0\} \). Suppose that \( x_1 \) is an arbitrary point in \( K \), let \( \{x_n\} \) and \( \{u_n\} \) be generated by

\[
\begin{align*}
& \left\{ \begin{array}{l}
    f(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \quad \forall \ y \in K, \\
    x_{n+1} = (1 - \alpha_n)u_n + \alpha_nSy_n, \\
    y_n = (1 - \beta_n)u_n + \beta_nSu_n,
\end{array} \right.
\end{align*}
\tag{12}
\]

where \( S = I - T \), \( \{\alpha_n\}, \{\beta_n\} \subset [0, 1) \) and \( r_n \) satisfy the following conditions:

(i) \( \alpha_n \leq \min\{k^2, \frac{1}{(2+k)(1-k)}, \frac{k(1-k)}{2L(1+L)}\} \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \); (ii) \( \beta_n \leq \frac{k(1-k)}{2L(1+L)}L \); (iii) \( r_n > 0 \).

Then \( \{x_n\} \) and \( \{u_n\} \) converge strongly to \( p \in \Omega \), respectively. Moreover, \( \{x_n\} \) admits error estimate

\[
\|x_n - p\| \leq \frac{1}{k} \sqrt{\exp^{-k^2 \sum_{i=1}^{n} \alpha_i}} \|x_1 - Tx_1\|. 
\]

**Proof.** Since \( N(T) = F(S) \), then Theorem 2.3 is true by theorem 2.1. Completing the proof of Theorem 2.3.

**Remark.** Since a contraction mapping must be a strong pseudo-contraction mapping, then the results in this paper are suitable for contraction mapping.

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