Abstract. Let $k$ an algebraically closed field and $R$ the homogeneous coordinate ring of $\mathbb{P}^n$ and $\Omega_{\mathbb{P}^n}$ the cotangent bundle of $\mathbb{P}^n$. In this paper I prove that for a given set $S$ of $s$ general points in $\mathbb{P}^n$ then the evaluation map $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(l)) \to \bigoplus_{i=1}^{s} \Omega_{\mathbb{P}^n}(l)|_{P_i}$ is of maximal rank. Implying that $a_0 = 0$ or $b_0 = 0$ so that $a_0b_0 = 0$ as conjectured by Anna Lorenzini [4, 5] see below

\[ \cdots \to R(-d - 2)^b_1 \oplus R(-d - 1)^a_0 \to R(-d - 1)^b_0 \oplus R(-d)^{(d+n)-s} \to I_S \to 0 \]

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1. INTRODUCTION

For a general set of points $\{P_1, \ldots, P_s\} \in \mathbb{P}^n$, with $s \geq n + 1$, then the homogeneous ideal of the sub-scheme of the union of these points, $I_S \subset R = k[x_0, \ldots, x_n]$, $k$ an algebraically closed field and $R$ the homogeneous coordinate ring of $\mathbb{P}^n$, has the following expected form:

\[ 0 \to F_{n-1} \to \cdots \to F_p \to \cdots \to F_0 \to I_S \to 0. \]
\[ F_p = R(-d-p)^{a_p-1} \bigoplus R(-d-p-1)^{b_p}, \]

\[ d \text{ being the smallest integer satisfying } s \leq h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)), \]

\[ a_p = \max \{ 0, h^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(d+p+1)) - \text{rk}(\Omega^p_{\mathbb{P}^n})s \}, \]

\[ b_p = \max \{ 0, \text{rk}(\Omega^p_{\mathbb{P}^n})s - h^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(d+p+1)) \}, \]

\[ (d+n-1) \choose n < s \leq (d+n) \choose n. \]

The problem can be reduced to showing the following; for all \( 0 \leq p \leq n-1 \) and non-negative integer \( l \) then existence of the above resolution is the same as saying the evaluation map below is of maximal rank i.e. it is surjective or injective or both; see [1].

\[ H^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(l)) \rightarrow \bigoplus_{i=1}^s \Omega^p_{\mathbb{P}^n}(l)|_{P_i}. \]

For this consider the exact sequence

\[ 0 \rightarrow \Omega_{\mathbb{P}^n}(1) \rightarrow W \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0 \]

Here, \( W = H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \), the set of linear forms and \( k[x_0, x_1, ..., x_n] = \text{Sym}(W) \)

Tensoring the sequence above with \( T_S(d) \) gives

\[ 0 \rightarrow T_S \otimes \Omega_{\mathbb{P}^n}(d+1) \rightarrow W \otimes T_S(d) \rightarrow T_S(d+1) \rightarrow 0 \]

Now taking global sections we get;

\[ 0 \rightarrow H^0(T_S \otimes \Omega_{\mathbb{P}^n}(d+1)) \rightarrow W \otimes I_d \rightarrow I_{d+1} \rightarrow H^1(T_S \otimes \Omega_{\mathbb{P}^n}(d+1)) \rightarrow 0 \]

Thus \( H^1(T_S \otimes \Omega_{\mathbb{P}^n}(d+1)) = I_{d+1}/W \cdot I_d \), corresponds to the minimal generators of \( I_S \) of degree \( d+1 \), and its dimension is \( b_0 \) i.e. \( h^1(T_S \otimes \Omega_{\mathbb{P}^n}(d+1)) = b_0. \)
Similarly, $H^0(T_S \otimes \Omega_{\mathbb{P}^n}(d+1))$ is the space of linear relations among the generators of degree $d$, whose dimension is $a_0$ i.e. $h^0(T_S \otimes \Omega_{\mathbb{P}^n}(d+1)) = a_0$.

Now consider the exact sequence

$$0 \longrightarrow T_S \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

Tensoring it by $\Omega_{\mathbb{P}^n}(d+1)$ gives:

$$0 \longrightarrow T_S \otimes \Omega_{\mathbb{P}^n}(d+1) \longrightarrow \Omega_{\mathbb{P}^n}(d+1) \longrightarrow \Omega_{\mathbb{P}^n}(d+1)|_S \longrightarrow 0$$

and now taking global sections yields

$$0 \longrightarrow H^0(T_S \otimes \Omega_{\mathbb{P}^n}(d+1)) \longrightarrow H^0(\Omega_{\mathbb{P}^n}(d+1)) \longrightarrow H^0(\Omega_{\mathbb{P}^n}(d+1)|_S)$$

$$\mu \longrightarrow H^1(T_S \otimes \Omega_{\mathbb{P}^n}(d+1))$$

$$\downarrow$$

$$H^1(\mathbb{P}^n \otimes \Omega_{\mathbb{P}^n}(d+1))$$

$$\downarrow$$

$$0$$

We will prove that $\mu$ is of maximal rank for a general set $S$ of $s$ points in $\mathbb{P}^n$.

As result, if $\mu$ is injective then its kernel is null i.e. $a_0 = h^0(T_S \otimes \Omega_{\mathbb{P}^n}(d+1)) = 0$ and the cokernel is not null that is $b_0 = h^1(T_S \otimes \Omega_{\mathbb{P}^n}(d+1))$ as expected. On other hand, if $\mu$ is surjective then we have the cokernel of $\mu$ being null i.e. $b_0 = h^1(T_S \otimes \Omega_{\mathbb{P}^n}(d+1)) = 0$ and the kernel of $\mu$ is not null that is, $a_0 = h^0(T_S \otimes \Omega_{\mathbb{P}^n}(d+1))$.

2. PRELIMINARIES

We use the statements (the so called *Enonces*) as in [1] by Hirschowitz and Simpson which F Lauze used in [2] to proof maximal rank for $T_{\mathbb{P}^n}$.

Let $X$ a smooth projective variety and $X'$ non-singular divisor of $X$. Let $F$ be a locally free sheaf on $X$ and

$$0 \longrightarrow F'' \longrightarrow F|_{X'} \longrightarrow F' \longrightarrow 0$$

be a exact sequence of locally free sheaves on $X'$. The kernel $E$ of $F \longrightarrow F'$ is a locally free sheaf on $X$ and we have another exact sequence of locally free sheaves on $X'$

$$0 \longrightarrow F'(-X') \longrightarrow E|_{X'} \longrightarrow F'' \longrightarrow 0$$

and as well exact sequences of coherent sheaves on $X$

$$0 \longrightarrow E \longrightarrow F \longrightarrow F' \longrightarrow 0$$
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and

\[
0 \longrightarrow F(-X) \longrightarrow E \longrightarrow F'' \longrightarrow 0.
\]

We have the following hypotheses:

\[ \text{R}(F, F', y; a, b, c) \]

\[ \text{RD}(F, F', y; a, b, c) \]

\[ \text{RD}(E, F'', y'; a', b', c') \]

2.1. **Notation.** Set \( X = \mathbb{P}^n, X' = \mathbb{P}^{n-1}, F = \Omega_{\mathbb{P}^n}, F' = \Omega_{\mathbb{P}^{n-1}}, E = \mathcal{O}_{\mathbb{P}^n}(-2), F'' = \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \).

The exact sequences of the elementary transformations after twisting by \( d + 1 \) are:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(d) & \longrightarrow & \Omega_{\mathbb{P}^n}(d) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(d-1)_{\oplus n} & \longrightarrow & \Omega_{\mathbb{P}^n}(d+1) & \longrightarrow & \Omega_{\mathbb{P}^{n-1}}(d+1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^{n-1}}(d) & \longrightarrow & \Omega_{\mathbb{P}^n}|_{\mathbb{P}^{n-1}}(d+1) & \longrightarrow & \Omega_{\mathbb{P}^{n-1}}(d+1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \\
0 & & 0 & & & & & & \\
\end{array}
\]

From which we have the hypotheses:

\[ H'_\Omega,n(d + 1; \alpha, \beta, \gamma) = H(\Omega_{\mathbb{P}^n}(d + 1), \Omega_{\mathbb{P}^{n-1}}(d + 1); \alpha, \beta, \gamma) \]

\[ H'_\mathcal{O,n}(d - 1; \rho, \sigma, \tau) = H(\mathcal{O}_{\mathbb{P}^n}(d - 1)_{\oplus n}, \mathcal{O}_{\mathbb{P}^{n-1}}(d); \rho, \sigma, \tau) \]

\[ H''_\mathcal{O,n}(d - 1; \rho, \sigma, \tau) = H(\mathcal{O}_{\mathbb{P}^n}(d - 1)_{\oplus n}, \mathcal{O}_{\mathbb{P}^{n-1}}(d); \rho, \sigma, \tau). \]

For the plane divisorial, with \( H \subseteq \mathbb{P}^n \) a hyperplane isomorphic to \( \mathbb{P}^{n-1} \) we shall utilize the sequence:

\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d - 2)_{\oplus n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d - 1)_{\oplus n} \longrightarrow \mathcal{O}_H(d - 1)_{\oplus n} \longrightarrow 0.
\]

**Hypothesis 2.1.** \( H'_{\Omega,n}(d + 1; \alpha, \beta, \gamma) \)

The hypothesis \( H'_{\Omega,n}(d + 1; \alpha, \beta, \gamma) \) asserts that for non-negative integers \( \alpha, \beta, \gamma \) and \( \varepsilon \) satisfying the conditions:

\[ 0 \leq \gamma \leq 1, \text{ and } 1 \leq \varepsilon \leq n - 2, \]

\[ n\alpha + n - 1\beta + \varepsilon\gamma = h^0(\Omega_{\mathbb{P}^n}(d + 1)), \text{ and} \]

\[ h^0(\Omega_{\mathbb{P}^n}(d + 1)). \]
(n − 1)β + εγ ≤ h^0(Ω_{P^{α−1}}(d + 1)) having for γ = 1 a quotient Γ' then the map

\[ η : H^0(P^n, Ω_{P^n}(d + 1)) \longrightarrow \bigoplus_{i=1}^{α} Ω_{P^n}(d + 1)|_{A_i} \oplus \bigoplus_{j=1}^{β} Ω_{P^{α−1}}(d + 1)|_{B_j} \oplus Γ'|_C \]

is bijective with h^0(Ω_{P^n}(d + 1)) = d(d + n) and for α general points A_1 \ldots A_α ∈ P^n, β + 1 general points B_1 \ldots B_β, C ∈ P^{α−1}.

**Hypothesis 2.2.** H_{Ω,n}(d + 1)

The hypothesis H_{Ω,n}(d + 1) asserts that H'_{Ω,n}(d + 1; α, β, γ) is true for all α, β and γ satisfying the conditions above.

**Hypothesis 2.3.** H'_{Ω,n}(d − 1; ρ, σ, τ)

The hypothesis H'_{Ω,n}(d − 1; ρ, σ, τ) asserts that for non-negative integers ρ, σ, τ and θ satisfying the conditions:

0 ≤ τ ≤ 1 and 2 ≤ θ ≤ n − 1,

nρ + σ + θτ = h^0(Ω_{P^n}(d − 1)^n), and

σ + θτ ≤ h^0(Ω_{P^{α−1}}(d)) having for τ = 1 a quotient Γ then the map

\[ φ : H^0(P^n, Ω_{P^n}(d − 1)^n) \longrightarrow \bigoplus_{i=1}^{ρ} Ω_{P^n}(d − 1)|_{R_i} \oplus \bigoplus_{j=1}^{σ} Ω_{P^{α−1}}(d)|_{S_j} \oplus Γ(S)|_T \]

is bijective with h^0(Ω_{P^n}(d − 1)^n) = n(d − 1) and for ρ general points R_1 \ldots R_ρ ∈ P^n, σ + 1 general points S_1 \ldots S_σ, T ∈ P^{α−1}.

**Hypothesis 2.4.** H_{Ω,n}(d − 1)

The hypothesis H_{Ω,n}(d − 1) asserts that H'_{Ω,n}(d − 1; ρ, σ, τ) is true for any ρ, σ, and τ satisfying the conditions above.

**Hypothesis 2.5.** H'_{Ω,n}(d − 1; ρ, σ, τ)

A variant version of the hypothesis H'_{Ω,n}(d − 1; ρ, σ, τ) with Γ independent of Γ' takes the form H'_{Ω,n}(d − 1; ρ, σ, τ) and it makes the same assertion as the hypothesis H'_{Ω,n}(d − 1; ρ, σ, τ) the only difference being quotient dependency.

### 3. THE METHODS OF HORACE

Méthode d’Horace simple[3] lemme 1

**Lemma 3.1.** Suppose we have a bijective morphism of vector spaces γ : H^0(X', F') → L and that we have H^1(X, E) = 0. Let μ : H^0(X, F) → L be a morphism of vector spaces. Then for H^0(X, F) → M ⊕ L to be of maximal rank it suffices that H^0(X, E) → M is of maximal rank.
**Lemma 3.2.** Suppose we are given a surjective morphism of vector spaces, 
\[ \lambda : H^0(\mathbb{P}^{n-1}, \Omega_{\mathbb{P}^{n-1}}(d+1)) \to L \] and suppose there exists a point \( Z' \in \mathbb{P}^{n-1} \) such that 
\[ H^0(\mathbb{P}^{n-1}, \Omega_{\mathbb{P}^{n-1}}(d+1)|_{Z'}) = 0. \] Then there exists a quotient \( \mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus n} \to \mathcal{D}(\lambda) \) with kernel contained in \( \Omega_{\mathbb{P}^{n-1}}(d+1)|_{Z'} \) of dimension \( \dim(\mathcal{D}(\lambda)) = \rho(\mathcal{O}_{\mathbb{P}^n}(d+1)) - \dim(\ker \lambda) \) having the following property. Let \( \mu : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d+1)) \to M \) be a morphism of vector spaces then there exists \( Z \in \mathbb{P}^{n-1} \) such that if \( H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus n}) \to M \oplus \mathcal{D}(\lambda) \) is of maximal rank then 
\[ H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d+1)) \to M \oplus L \oplus \Omega_{\mathbb{P}^n}(d+1)|_{Z} \] is also of maximal rank.

The sequences for the quotient are as follows:

\[
\begin{array}{cccc}
& 0 & 0 \\
\downarrow & & & \\
\dim n - 1 & \Omega_{\mathbb{P}^{n-1}}(d)|_{Z} & \to & \mathcal{D}'|_{Z} \\
\downarrow & & & \downarrow \\
\dim n & \mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus n}|_{Z} & \to & \mathcal{D}|_{Z} \cong \mathcal{O}_{\mathbb{P}^{n-1}}|_{Z} \oplus \mathcal{D}'|_{Z} \\
\downarrow & & & \downarrow \\
\dim 1 & \mathcal{O}_{\mathbb{P}^{n-1}}(d)|_{Z} & \to & \mathcal{O}_{\mathbb{P}^n}(d)|_{Z} \\
\downarrow & & & \downarrow \\
& 0 & 0 \\
\end{array}
\]

\[ \dim n - 3 \ (n - 2) \]

\[ \dim n - 2 \ (n - 1) \]

\[ \dim 1 \]

3.1. **The Vectorial Methods.**

**Lemma 3.3.** **Vectorial Method 1**

Let \( \alpha, \beta, \gamma, d \) and \( \varepsilon \) be non-negative integers satisfying the conditions of Hypothesis 2.1 and \( \rho, \sigma, \tau \) and \( \theta \) non-negative integers satisfying the conditions of Hypothesis 2.3 then the Hypothesis \( H'_{\mathcal{O}, n}(d-1; \rho, \sigma, \tau) \) implies \( H'_{\mathcal{O}, n}(d+1; \alpha, \beta, \gamma) \).

**Proof.** Consider the exact sequence:

\[ 0 \to \mathcal{O}_{\mathbb{P}^n}(d-1)^{\oplus n} \to \Omega_{\mathbb{P}^n}(d+1) \to \Omega_{\mathbb{P}^{n-1}}(d+1) \to 0 \]

and let \( B \) and \( C \) be general subsets of \( \mathbb{P}^{n-1} \). We specialize \( A \) to \( R \cup S \cup T \) with \( R \) a general set of \( \rho \) points in \( \mathbb{P}^n \) and \( S \) and \( T \) sets of \( \sigma \) and \( \tau \) general points in \( \mathbb{P}^{n-1} \). To run points to \( \mathbb{P}^{n-1} \), consider the map, \( \gamma : H^0(\Omega_{\mathbb{P}^{n-1}}(d+1)) \to H^0(\Omega_{\mathbb{P}^{n-1}}(d+1)|_{\partial B}) \oplus \Gamma'_{\partial C} \), if
the number of points we have satisfy \( h^0(\Omega_{\mathbb{P}^{n-1}}(d+1)) \) then \( \gamma \) is bijective, if not then we specialize as many more points as we need to \( \mathbb{P}^{n-1} \) in order for \( \gamma \) to become bijective.

Taking global sections for the exact sequence above and evaluating we construct;

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\uparrow & & \uparrow \\
H^0(\Omega_{\mathbb{P}^{n-1}}(d+1)) & \overset{\gamma}{\longrightarrow} & H^0(\Omega_{\mathbb{P}^{n-1}}(d+1)|_{B \cup S}) \oplus \Gamma'_{|c} \oplus \Gamma_{|T} \\
\uparrow & & \uparrow \\
H^0(\Omega_{\mathbb{P}^{n}}(d+1)) & \overset{\beta}{\longrightarrow} & H^0(\Omega_{\mathbb{P}^{n}}(d+1)|_{R \cup S \cup T = A}) \oplus H^0(\Omega_{\mathbb{P}^{n-1}}(d+1)|_{B}) \oplus \Gamma'_{|c} \\
\uparrow & & \uparrow \\
H^0(\Omega_{\mathbb{P}^{n}}(d-1)^{\oplus n}) & \overset{\alpha}{\longrightarrow} & H^0(\Omega_{\mathbb{P}^{n}}(d-1)^{\oplus n}|_{R}) \oplus H^0(\Omega_{\mathbb{P}^{n-1}}(d)|_{S}) \oplus \Gamma_{|T} \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}
\]

From the above diagram of exact sequences, by Inductive hypothesis on \( \mathbb{P}^{n-1} \) and Lemma 3.2 the map \( \gamma \) is bijective and hence if \( \alpha \) is bijective then \( \beta \) is bijective as well and this gives \( H'_{\alpha,\rho}(d-1; \rho, \sigma, \tau) \) implies \( H'_{\alpha,\rho}(d+1; \alpha, \beta, \gamma) \)

\[\square\]

**Lemma 3.4. Vectorial Method 2**

Let \( \rho, \sigma, \tau \) and \( \theta \) non-negative integers satisfying the conditions of Hypothesis 2.3 and \( \overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\tau} \) be non-negative integers satisfying conditions similar to those of Hypothesis 2.1 with the Hypothesis \( H'_{\overline{\alpha},\gamma}(d; \overline{\alpha}, \overline{\beta}, \overline{\gamma}) \) being the same as Hypothesis 2.1 but twisted by 1, then the Hypothesis \( H'_{\overline{\alpha},\rho}(d; \overline{\alpha}, \overline{\beta}, \overline{\tau}) \) implies \( H'_{\overline{\alpha},\rho}(d-1; \rho, \sigma, \tau) \).

**Proof.** Consider the exact sequence;

\[
0 \longrightarrow \Omega_{\mathbb{P}^{n}}(d) \longrightarrow \Omega_{\mathbb{P}^{n}}(d-1)^{\oplus n} \longrightarrow \Omega_{\mathbb{P}^{n-1}}(d) \longrightarrow 0
\]

and let \( S \) and \( T \) general sets of \( \sigma \) and \( \tau \) points in \( \mathbb{P}^{n-1} \), specialize \( R \) to \( A \cup B \), where \( A \) is a general set of \( \overline{\alpha} \) points in \( \mathbb{P}^{n} \) and \( B \) is a general set of \( \overline{\beta} \) points in \( \mathbb{P}^{n-1} \) with \( C = T \).

Now consider the evaluation map, \( \gamma : H^0(\Omega_{\mathbb{P}^{n-1}}(d)) \longrightarrow H^0(\Omega_{\mathbb{P}^{n-1}}(d)|_{S \cup T}) \), if the number of points we have are enough to satisfy \( h^0(\Omega_{\mathbb{P}^{n}}(d)) \) then \( \overline{\gamma} \) bijective, if not then we specialize as many more points, \( \overline{\beta} \), in this case, to \( \mathbb{P}^{n-1} \) in order for \( \overline{\gamma} \) to become bijective.

Taking global sections for the exact sequence above and evaluating at corresponding points we construct a diagram of exact sequences as follows;
The map $\gamma$ is bijective giving the Hypothesis $H'_{\Omega,n}(d;\rho,\sigma,\tau)$ implies $H'_{\Omega,n}(d-1;\rho,\sigma,\tau)$. When the number of points we have in $P^{n-1}$ are few relative to $d$ we use the plane divisorial method in preference to this method.

**Lemma 3.5. Plane Divisorial**

Let $\rho, \sigma, \tau$ and $\theta$ non-negative integers satisfying the conditions of Hypothesis 2.3 and set $\rho' = \rho - h^0(\mathcal{O}_{P^{n-1}}(d-1))$. If $\rho' \geq 0$ and $\sigma + \tau \leq h^0(\mathcal{O}_{P^{n-1}}(d-1))$ then the Hypothesis $H_{\Omega,n}(d-2;\rho',\sigma,\tau)$ implies $H_{\Omega,n}(d-1;\rho,\sigma,\tau)$.

**Proof.** Let $R$ be a general set of $\rho$ points in $P^n$, $S$ and $T$ be general sets of $\sigma$ and $\tau$ points in $P^{n-1}$ such that they are fewer relative to $d$ (i.e. when Vectorial Method 2 fails). We choose a hyperplane $H \subset P^n$ disjoint from $S$ and $T$ with $H \cong P^{n-1}$ and specialize $\rho'$ points from $P^n$ to $H$ (i.e. $R'$ is the set we have after specializing from $R$ in $P^n$) so that $H^0(H,\mathcal{O}_H(d-1)^{\oplus n}) \longrightarrow H^0(\mathcal{O}_H(d-1)^{\oplus n}_{R'})$ is bijective that is set $\rho - \rho' = h^0(\mathcal{O}_{P^{n-1}}(d-1))$ and so taking global sections for the sequence

\[
0 \longrightarrow \mathcal{O}_{P^n}(d-2)^{\oplus n} \longrightarrow \mathcal{O}_{P^n}(d-1)^{\oplus n} \longrightarrow \mathcal{O}_H(d-1)^{\oplus n} \longrightarrow 0
\]
we construct a diagram of exact sequences:

\[
\begin{array}{ccc}
H^0(H, \mathcal{O}_H(d-1)^{\otimes n}) & \xrightarrow{\alpha} & H^0(\mathcal{O}_H(d-1)^{\otimes n}) \\
\uparrow & & \uparrow \\
H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)^{\otimes n}) & \xrightarrow{\beta} & H^0(\mathcal{O}_{\mathbb{P}^n}(d-1)^{\otimes n}) \\
\uparrow & & \uparrow \\
H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-2)^{\otimes n}) & \xrightarrow{\gamma} & H^0(\mathcal{O}_{\mathbb{P}^n}(d-2)^{\otimes n}) \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}
\]

Since \(\alpha\) is bijective then \(\gamma\) bijective implies \(\beta\) is also bijective and this gives the Hypothesis

\(H_{0,n}(d-2; \rho', \sigma, \tau)\) implies \(H_{0,n}(d-1; \rho, \sigma, \tau)\).

\[\blacksquare\]

3.2. Hypercritical méthode d’Horace.

**Lemma 3.6.** Consider \(H'_{0,n}(d-1; s_1, s_2, 0)\) with \(d \geq 1, s_1, s_2\) being non-negative integers that satisfy: \(ns_1 + s_2 = h^0(\mathcal{O}_{\mathbb{P}^n}(d-1)^{\otimes n})\) and \(s_2 \leq h^0(\mathcal{O}_{\mathbb{P}^n}(d))\). Now suppose that the \(H^0(\Omega_{\mathbb{P}^n}(d)) \rightarrow H^0(\Omega_{\mathbb{P}^n}(d)|_{S_1})\) is injective and \(H^0(\mathcal{O}_{\mathbb{P}^n}(d-1)^{\otimes n}) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(d-1)^{\otimes n}|_{S_1})\) is surjective with a general \(S_1 \subseteq \mathbb{P}^n\) then the Hypothesis \(H'_{0,n}(d-1; s_1, s_2, 0)\) is true.

This Lemma is for when we have no quotient.

**Proof.** See [6] Lemma 1.11. \[\blacksquare\]

**Lemma 3.7.** Consider \(H'_{0,n}(d-1; s_1, s_2, 1)\) where \(d \geq 1, s_1, s_2\) and \(2 \leq \theta \leq n-1\) are non-negative integers such that, \(ns_1 + s_2 + \theta = h^0(\mathcal{O}_{\mathbb{P}^n}(d-1)^{\otimes n})\) and \(s_2 + \theta \leq h^0(\mathcal{O}_{\mathbb{P}^n}(d))\). Under the same Hypotheses as Lemma 2.1 i.e. \(H^0(\Omega_{\mathbb{P}^n}(d)) \rightarrow H^0(\Omega_{\mathbb{P}^n}(d)|_{S_1})\) is injective and \(H^0(\mathcal{O}_{\mathbb{P}^n}(d-1)^{\otimes n}) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(d-1)^{\otimes n}|_{S_1})\) is surjective then the Hypothesis \(H''_{0,n}(d-1; s_1, s_2, 1)\) is true.

**Proof.** See [6] Lemma 1.12. \[\blacksquare\]

3.3. The Main Theorem.

**Theorem 3.8.** Suppose \(H_{\Omega,n}(d+1)\) is true. Then for any non-negative integer \(m\), there exists a set, \(M = \{P_1, P_2, \ldots, P_m\}\) of \(m\) points in \(\mathbb{P}^n\) such that the evaluation map, \(\mu\), is of maximal rank.

\[\mu : H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(d+1)) \rightarrow \bigoplus_{i=1}^{m} \Omega_{\mathbb{P}^n}(d+1)|_{P_i}\]
Proof. (a) If $h^0(\Omega_{P^n}(d + 1)) \equiv 0 \pmod{n}$ then $r$ is the critical number of points needed for bijectivity i.e. the map $H^0(P^n, \Omega_{P^n}(d + 1)) \rightarrow \bigoplus_{i=1}^{r} \Omega_{P^n}|_{P_i}$ is bijective. Set $\pi = \lfloor \frac{1}{n} h^0(\Omega_{P^n}(d + 1)) \rfloor$

we now have the following cases:

(i) if $m = r$ then our map is bijective since we have the same number of points as the critical number i.e. the map $\alpha$ is bijective and $\gamma$ an identity map and so $\mu$ is bijective see below:

$$\begin{array}{ccc}
H^0(P^n, \Omega_{P^n}(d + 1)) & \xrightarrow{\mu} & \bigoplus_{i=1}^{r} \Omega_{P^n}|_{P_i} \\
\downarrow{\alpha} & & \uparrow{\gamma} \\
\bigoplus_{i=1}^{n} \Omega_{P^n}|_{P_i} \oplus \bigoplus_{i=n+1}^{r} \Omega_{P^n}|_{P_i} & & \\
\end{array}$$

(ii) if $m > r$ i.e. we have more points than the critical number and our map is injective i.e. since $\alpha$ is bijective and $\gamma$ surjective then our map $\mu$ has to inject see below:

$$\begin{array}{ccc}
H^0(P^n, \Omega_{P^n}) & \xrightarrow{\alpha} & \bigoplus_{i=1}^{r} \Omega_{P^n}|_{P_i} \\
\downarrow{\alpha} & & \uparrow{\gamma} \\
\bigoplus_{i=1}^{r} \Omega_{P^n}|_{P_i} \oplus \bigoplus_{i=r+1}^{m} \Omega_{P^n}|_{P_i} & & \\
\end{array}$$

(iii) if $m < r$ then we have the less points than the critical number thus our map surjects i.e. since $\alpha$ is bijective and $\gamma$ surjective then our map $\mu$ is surjective.

References


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