Fixed Point Theorems for a $k$-set Contraction Map on a Nearly-convex Subset of a Locally Convex Space$^1$

Chi-Ming Chen and Cheng-Te Liu

Department of Applied Mathematics
National Hsin-Chu University of Education, Taiwan, R.O.C.

Abstract

In this paper, the first part, we establish the fixed point theorems for a $k$-set contraction map on the class $Q(X,Y)$, (see [1]) . The second part, we generalize the $KKM$ property on a convex set [2] to the $KKM^*$ property on a nearly-convex set, and then we establish fixed point theorems, matching theorems and variational inequalities for this class.

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1 Introduction and Preliminaries

In 1930, Schauder had shown that a continuous compact map $f : X \to X$ defined on a closed convex subset $X$ of a Banach space has a fixed point. This theorem enormous influence on fixed point theory, variational inequalities and equilibrium problems. Recently there are appeared some results on fixed point so-called Kakutani factorizable multi-functions defined on convex sets. In this paper, we will invoke nearly-convexity of constraint regions in place of convexity. The first part, we establish the fixed point theorem and coincidence theorem for a $k$-set contraction map on the class $Q(X,Y)$. The second part, we generalize the $KKM$ property on a convex set to the $KKM^*$ property on a nearly-convex set, and then we establish the fixed point theorem for a $k$-set contraction map on the family $KKM^*(X,X)$, which not need a compact map.

$^1$Research supported by the NSC.
Let $X$ and $Y$ be two sets, and let $T : X \to 2^Y$ be a set-valued mapping. We shall use the following notations in the sequel.

(i) $T(x) = \{ y \in Y : y \in T(x) \}$,
(ii) $T(A) = \cup_{x \in A} T(x)$,
(iii) $T^{-1}(y) = \{ x \in X : y \in T(x) \}$,
(iv) $T^{-1}(B) = \{ x \in X : T(x) \cap B \neq \phi \}$,
(v) $T$ is said to lower semicontinuous if for each open subset $B$ of $Y$, $T^{-1}(B)$ is open in $X$, and
(vi) if $D$ is a nonempty subset of $X$, then $\langle D \rangle$ denote the class of all nonempty finite subset of $D$.

For the case that $X$ and $Y$ are two topological spaces. Then $T$ is said to be closed if its graph $\mathcal{G}_T = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed. $T$ is said to be compact if the image $T(X)$ of $X$ under $T$ is contained in a compact subset of $Y$. A subset $D$ of $X$ is said to be compactly closed (resp. compactly open) in $X$ if for any compact subset $K$ of $X$, the set $D \cap K$ is closed (resp. closed) in $K$. Obviously, $D$ is compactly open in $X$ if and only if its complement $D^c$ is compactly closed in $X$.

We now introduce a new class of nearly-convex sets. A nonempty subset $X$ of a Hausdorff topological vector space $E$ is said to be nearly-convex [4] if for every compact subset $A$ of $X$ and every neighborhood $V$ of the origin $0$ of $E$, there is a continuous mapping $h_{A,V} : A \to X$ such that $x \in h_{A,V}(x) + V$ for each $x \in A$ and $co(h_{A,V}(A)) \subset X$. We call $h_{A,V}$ a continuous convex-inducing mapping.

**Remark 1** (i) In general, the continuous convex-inducing mapping $h_{A,V}$ is not unique. If $U \subset V$, then it is clear that any $h_{A,U}$ can be regarded as an $h_{A,V}$.

(ii) It is clear that the convex set is nearly-convex, but the inverse is not true. For a counterexample, let $(M,d)$ be a metric space, $M = \mathbb{R}^2$, we define the metric $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $x = (x_1, x_2)$, $y = (y_1, y_2) \in M$. Then the set $B(0) = \{ x \in M : d(x,0) < 1 \} \cup \{(1,0), (0,1)(0, -1)\}$ is a nearly-convex set, but not convex.

Let $E$ denote a Hausdorff topological vector space, and $B(E)$ the family of nonempty bounded subsets.
Let $\mathcal{P} = \{P|P\text{ is a family of seminorms which determines the topology on } E\}$. Let $\mathcal{R}^+$ be the set of all nonnegative real numbers. A mapping $\Phi : B(E) \to \mathcal{R}^+$ is called a measure of noncompactness [3] provided the following conditions hold:

(i) $\Phi(\overline{co}(\Omega)) = \Phi(\Omega)$ for each $\Omega \in B(E)$, where $\overline{co}(\Omega)$ denotes the closure of the convex hull of $\Omega$,

(ii) $\Phi(\Omega) = 0$ if and only if $\Omega$ is precompact,

(iii) $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$, for each $A, B \in B(E)$, and

(iv) $\Phi(\lambda \Omega) = \lambda \Phi(\Omega)$, for each $\lambda \geq 0$, $\Omega \in B(E)$.

The above notion is a generalization of the set measure of noncompactness; if $\{p : P\}, P \in \mathcal{P}$ is a family of seminorms which determines the topology on $E$, then for each $p \in P$ and $\Omega \in E$, we define the set-measure of noncompactness $\alpha_p : 2^E \to \mathcal{R}^+$ by

$$\alpha_p(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ can be covered by a finite number of sets and each}$$

$$p - \text{diameter of the sets is less than } \varepsilon \}$$

where the $p - \text{diameter of a set } D = \sup\{p(x - y) : x, y \in D\}$.

**Definition 1** A mapping $T : X \to 2^E$ is said to be a $k$-set contraction map, if there exists $P \in \mathcal{P}$ such that for each $p \in P$, $\alpha_p(T(\Omega)) \leq k\alpha_p(\Omega)$ with $k \in (0, 1)$ for each bounded subset $\Omega$ of $X$ and $T(X)$ is bounded.

Let $X$ be a subset of a Hausdorff topological vector space $E$ and $Y$ a topological space, we now define a new class $Q(X, Y)$ of the set-valued mappings from $X$ into $Y$ as follows:

$T \in Q(X, Y)$

$\iff$ for any compact convex subset $K$ of $X$ and any continuous mapping

$$f : T(K) \to K, \text{ the composition } f(T|_K) : K \to K \text{ has a fixed point.}$$

We next generalize the $KKM$ property (see, [2]) on a convex set of a topological vector space to the following form for a nearly-convex set $X$. 

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We next generalize the $KKM$ property (see, [2]) on a convex set of a topological vector space to the following form for a nearly-convex set $X$. 

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Definition 2 Let $X$ be a nearly-convex subset of a topological vector space $E$, and $Y$ a topological space. If $T, F : X \to 2^Y$ are two set-valued mappings such that for each finite subset $A$ of $X$ and every neighborhood $V$ of the origin 0 of $E$, there exists a continuous convex-inducing mapping $h_{A,V} : A \to X$ such that $T(co(h_{A,V}(A))) \subset F(A)$, then we call $F$ a generalized $KKM^*$ mapping with respect to $T$.

Let $T : X \to 2^Y$ be a set-valued mapping such that if $F : X \to 2^Y$ is a generalized $KKM^*$ mapping with respect to $T$ then the family $\{F_x : x \in X\}$ has the finite intersection property, then we say that $T$ has the $KKM^*$ property. Denote

$$KKM^*(X,Y) = \{T : X \to 2^Y \mid T \text{ has the } KKM^* \text{ property}\}.$$ 

Remark 2 In particular, for the case $co(A) \subset X$, we may let the convex-inducing mapping $h_{A,V}$ be the identity mapping.

Remark 3 In general, $Q(X,Y)$ and $KKM^*(X,Y)$ may not be comparable. (see,[1])

2 Fixed point theorem for the class $Q(X,Y)$

In this section, we establish a fixed point theorem for a $k$-set contraction map on the class $Q(X,Y)$, which not need to be a compact map.

Theorem 1 Let $X$ be a nonempty bounded nearly-convex subset of a Hausdorff topological vector space $E$. Assume that $T$ is a $k$-set contraction map, $0 < k < 1$. Then $X$ contains a precompact nearly-convex subset.

Proof. Since $T$ is a $k$-set contraction map, $0 < k < 1$, there exists $P \in \mathcal{P}$ such that for each $p \in P$, we have $\alpha_p(T(A)) \leq k\alpha_p(A)$ for each subset $A$ of $X$. Take $y \in X$. Let

$$X_0 = X, \quad X_1 = co(T(X_0) \cup \{y\}) \cap X, \quad \text{and}$$

$$X_{n+1} = co(T(X_n) \cup \{y\}) \cap X, \quad \text{for each } n \in N.$$ 

Then,

1. $X_n$ is nearly-convex, for each $n \in N$,
2. $X_{n+1} \subset X_n$, for each $n \in N$
3. $T(X_n) \subset X_{n+1}$, for each $n \in N$ and
4. $\alpha_p(X_{n+1}) \leq \alpha_p(T(X_n)) \leq k\alpha_p(X_n) \leq \ldots \leq k^{n+1} \alpha_p(X_0)$, for each $n \in N$. 


Thus $\alpha_p(X_n) \to 0$, as $n \to \infty$, and hence $X_\infty = \bigcap_{n \geq 1} X_n$ is a nonempty precompact nearly-convex set. □

**Remark 4** In the process of the proof of Theorem 1, we call the set $X_\infty$ a precompact-inducing nearly-convex subset of $X$.

**Corollary 1** Let $X$ be a nonempty bounded convex subset of a Hausdorff topological vector space $E$. Assume that $T$ is a $k$-set contraction map, $0 < k < 1$. Then $X$ contains a precompact-inducing convex subset.

**Theorem 2** Let $X$ be a nonempty bounded nearly-convex subset of a Hausdorff topological vector space $E$, and let $T \in Q(X, X)$. Assume that:

(i) $T$ is a $k$-set contraction map, $0 < k < 1$ and closed, and

(ii) the precompact-inducing nearly-convex subset $X_\infty$ of $X$, $T(X_\infty) \subset X_\infty$

Then $T$ has a fixed point in $X$.

*Proof.* Let $\mathcal{N} = \{U_i : i \in I\}$ be a local base of $E$ such that $U_i$ is symmetric and open for each $i \in I$, and let $V \in \mathcal{N}$.

By (i), since $\alpha_p(T(X_n)) \to 0$, as $n \to \infty$, hence $\overline{T(X_\infty)} = \bigcap_{n \geq 1} \overline{T(X_n)}$ is a nonempty compact subset of $X_\infty$. And, since $X_\infty$ is nearly-convex, there is a continuous mapping $h_{T(X_\infty), V} : \overline{T(X_\infty)} \to X_\infty$ such that $x \in h_{T(X_\infty), V}(x) + V$ for each $x \in \overline{T(X_\infty)}$ and $\text{co}(h_{T(X_\infty), V}(T(X_\infty))) \subset X_\infty$.

Let $Z = \text{co}(h_{T(X_\infty), V}(T(X_\infty)))$. Then $Z$ is a compact and convex subset of $X$, and $T(Z) \subset X_\infty$. Since $T \in Q(X, X)$ and $h_{T(X_\infty), V}$ is continuous, the composition $h_{T(X_\infty), V} \circ T|Z : Z \to 2^Z$ has a fixed point, say $x_V \in h_{T(X_\infty), V}(T(x_V))$. Let $x_V = h_{T(X_\infty), V}(y_V)$ for some $y_V \in T(x_V) \subset T(Z) \subset T(X_\infty)$. Then $y_V \in x_V + V = h_{T(X_\infty), V}(y_V) + V$. Since $\overline{T(X_\infty)}$ is compact, we may assume that $\{y_V\}$ converges to $\overline{x}$, and then $\{x_V\}$ also converges to $\overline{x}$. The closedness of $T$ implies $\overline{x} \in T(\overline{x})$. □

Follow above Theorem 2, we immediately have the following corollary.

**Corollary 2** [1] Let $X$ be a nonempty nearly-convex subset of a Hausdorff topological vector space $E$, and let $T \in Q(X, X)$ be compact and closed with $T(X) \subset X$. Then $T$ has a fixed point in $X$.

We next establish the following coincidence theorem.
Theorem 3 Let \( X \) be a nonempty bounded convex subset of a Hausdorff topological vector space \( E \), and let \( T, G : X \to 2^X \) be two set-valued mappings. Assume that:

(i) \( T \in Q(X,X) \) is a \( k \)-set contraction map, \( 0 < k < 1 \) and closed with \( \overline{T(X)} \subset X \),

(ii) for each \( y \in G(X) \), \( G^{-1}(y) \) is convex, and

(iii) for the precompact-inducing convex subset \( X_\infty \) of \( X \), \( \overline{T(X_\infty)} \subset \bigcup \{ \text{int}G(x) : x \in X_\infty \} \).

Then there exists an \( x_0 \in X \) such that \( T(x_0) \cap G(x_0) \neq \phi \).

Proof. By the same process of the proof of Theorem 2, we get a compact subset \( \overline{T(X_\infty)} \) of \( X \).

By (iii), there exists a finite subset \( \{ x_1, x_2, \ldots, x_n \} \) of \( X_\infty \) such that \( \overline{T(X_\infty)} \subset \bigcup_{i=1}^n \text{int}G(x_i) \). Let \( \{ \lambda_i \}_{i=1}^n \) be a partition of the unity subordinated to \( \{ \text{int}G(x_i) : i = \{1,2,\ldots,n\} \} \), and let \( P = \text{co}\{x_1, x_2, \ldots, x_n\} \). Define \( f : \overline{T(X_\infty)} \to P \) by

\[
f(y) = \sum_{i=1}^n \lambda_i(y)x_i = \sum_{i \in N_y} \lambda_i(y)x_i, \text{ for each } y \in \overline{T(X_\infty)},
\]

where \( i \in N_y \) iff \( \lambda_i(y) \neq 0 \) iff \( y \in \text{int}G(x_i) \subset G(x_i) \).

Then \( x_i \in G^{-1}(y) \) for each \( i \in N_y \). Clearly, \( f \) is continuous, and by (ii), we have

\[
f(y) \in \text{co}\{x_i : i \in N_y\} \subset G^{-1}(y), \text{ for each } y \in \overline{T(X_\infty)}.
\]

Since \( P \) is compact convex subset of \( X \) and \( T \in Q(X,X) \), the composition \( f|T(P) \circ T|_P : P \to P \) has a fixed point \( x_0 \in P \subset X \). So \( x_0 \in fT(x_0) \) and \( f^{-1}(x_0) \subset G(x_0) \). So, we complete the proof. \( \Box \)

3 Fixed point theorem for the class \( KKM^*(X,Y) \)

In this section, we establish the fixed point theorem for a \( k \)-set contraction map on the class \( KKM^*(X,Y) \), which not need to be a compact map.

The following Lemma 1 will play important role for this paper.

Lemma 1 Let \( X \) be a nonempty nearly-convex subset of a Hausdorff topological vector space \( E \), \( Y \) a topological space. Then \( T|_D \in KKM^*(D,Y) \) whenever \( T \in KKM^*(X,Y) \) and \( D \) is a nonempty nearly-convex subset of \( X \).
Proof. Let $F : D \to 2^Y$ be a generalized $KKM^*$ mapping with respect to $T|_D$. Then for any finite subset $A$ of $D$ and any neighborhood $V$ of the origin 0 of $E$, there exists a continuous convex-inducing mapping $h_{A,V} : A \to D$ such that $T|_D(co(h_{A,V}(A))) \subset F(A)$.

Define $F' : X \to 2^Y$ by

$$F'(x) = \begin{cases} F(x) & x \in D, \\ Y & x \in X \setminus D. \end{cases}$$

It is clear that for any finite subset $B$ of $X$ and any any neighborhood $U$ of the origin 0 of $E$, there exists a continuous convex-inducing mapping $h_{B,U} : B \to X$ such that $T(co(h_{B,U}(B))) \subset F'(B)$. Indeed,

1. if $B \not\subseteq D$, then there exists some $x \in B \setminus D$, and hence $F'(x) = Y$. So the result is obvious.
2. if $B \subseteq D$, since $F$ be a generalized $KKM^*$ mapping with respect to $T|_D$, the inclusion is true.

Thus $F'$ is a generalized $KKM^*$ mapping with respect to $T$. Since $T \in KKM^*(X,Y)$, hence the family $\{F'x : x \in X\}$ has finite intersection property, and so does the family $\{\overline{T}x : x \in D\}$. So $T|_D \in KKM^*(D,Y)$. $\square$

**Lemma 2** Let $X$ be a nonempty nearly-convex subset of a Hausdorff topological vector space $E$, $Y$ and $z$ are two topological spaces. Then $fT \in KKM^*(X,Z)$ whenever $T \in KKM^*(X,Y)$ and $f \in C(Y,Z)$.

**Proof.** Let $F$ be a generalized $KKM^*$ mapping with respect to $fT$. Then for any $A = \{x_1, x_2, \ldots, x_n\} \in \langle X \rangle$ and any neighborhood of the origin 0 of $E$, there exists a continuous convex-inducing mapping $h_{A,V}(A) : A \to X$ such that $fT(co(h_{A,V}(A))) \subset \bigcup_{i=1}^n F(x_i)$. So, $T(co(h_{A,V}(A))) \subset \bigcup_{i=1}^n f^{-1}F(x_i)$, which says that $f^{-1}F$ is a generalized $KKM^*$ mapping with respect to $T$. Since $T \in KKM^*(X,Y)$, the family $\{f^{-1}F(x) : x \in X\}$ has the finite intersection property, and so does the family $\{F(x) : x \in X\}$. This shows that $fT \in KKM^*(X,Z)$. $\square$

**Theorem 4** Let $X$ be a nonempty bounded nearly-convex subset of a locally convex space $E$, and let $T \in KKM^*(X,X)$ is a $k$-set contraction, $0 < k < 1$ and closed with $\overline{T(X)} \subset X$. Then $T$ has a fixed point in $X$.

**Proof.** Let $\mathcal{N} = \{U_i : i \in I\}$ be a local base of $E$ such that $U_i$ is symmetric, open and convex for each $i \in I$, and let $V \in \mathcal{N}$.

And, by the same process of the proof of Theorem 2, we get a compact subset $\overline{T(X_\infty)}$ of $X$. Since $\overline{T(X_\infty)} \subset T(X) \subset X$ and $X$ is nearly-convex, there
is a continuous mapping $h_{\overline{T(X_\infty)},V} : \overline{T(X_\infty)} \to X$ such that $x \in h_{\overline{T(X_\infty)},V}(x) + V$ for each $x \in \overline{T(X_\infty)}$ and $\text{co}(h_{\overline{T(X_\infty)},V}(\overline{T(X_\infty)})) \subset X$.

Let $Z = \text{co}(h_{\overline{T(X_\infty)},V}(\overline{T(X_\infty)}))$. Then $Z$ is a compact and convex subset of $X$, and $T(Z) \subset X$. Since $T \in KKM^*(X,X)$ and $X_\infty$ is a nearly-convex subset of $X$, by Lemma 1, we have $T|_{X_\infty} \in KKM^*(X_\infty,X)$. Next, since $h_{\overline{T(X_\infty)},V}$ is continuous, by Lemma 2, we have $h_{\overline{T(X_\infty)},V} \circ T|_{X_\infty} \in KKM^*(X_\infty,Z)$. Put $F = h_{\overline{T(X_\infty)},V} \circ T|_{X_\infty}$. Then $F$ is closed, since $T$ is closed, $h_{\overline{T(X_\infty)},V}$ is continuous and $\overline{T(X_\infty)}$ is compact.

We now claim that for each $U_i \in \mathcal{N}$, there exists an $x_i \in X_\infty$ such that

$$(x_i + U_i + V) \cap F(x_i) \neq \phi.$$ 

If the above statement is not true, then there exists $U \in \mathcal{N}$ such that $(x + U + V) \cap F(x) = \phi$, for all $x \in X_\infty$.

Let $K = \overline{h_{\overline{T(X_\infty)},V}(T(X_\infty))}$. Then $K \subset \overline{h_{\overline{T(X_\infty)},V}(T(X_\infty))} \subset Z$. Define $G : X_\infty \to 2^Z$ by

$$G(x) = K \setminus (x + \frac{1}{2}U + V), \quad \text{for each } x \in X_\infty.$$ 

Then

1. $G(x)$ is compact, for each $x \in X_\infty$, and
2. $G$ is a generalized $KKM^*$ mapping with respect to $F$.

We prove (2) by contradiction. Let $A = \{x_1, x_2, ..., x_n\} \subset X_\infty$. Then there exists $V' \in \mathcal{N}$ such that for any continuous convex-inducing $h_{A,V'} : A \to X_\infty$ one has $F(\text{co}(h_{A,V'}(A))) \not\subset G(A)$. Let $U' \in \mathcal{N}$ such that $U' \subset \frac{1}{2}U \cap V'$. Then $F(\text{co}(h_{A,U'}(A))) \not\subset G(A)$. So, there is $u \in \text{co}(h_{A,U'}(A))$ and $v \in F(u)$ such that $v \notin \bigcup_{i=1}^m G(x_i)$.

From the definition of $G$, it follows that $v \in (x_i + \frac{1}{2}U + V)$, for each $i \in \{1, 2, ..., n\}$. Hence $v \in (x_i + \frac{1}{2}U + V \cap V + U') \subset (h_{A,U'}(x_i) + \frac{1}{2}U + V) \subset (h_{A,U'}(x_i) + U + V)$, for each $i \in \{1, 2, ..., n\}$, since $X_\infty$ is nearly-convex. Thus, $h_{A,U'}(x_i) \in (v + U + V)$, for each $i \in \{1, 2, ..., n\}$, and hence $u \in \text{co}(h_{A,U'}(A)) \subset v + U + V$. So, $v \in u + U + V$ and $v \in F(u)$, we conclude that $F(u) \cap (u + U + V) \neq \phi$, a contradiction. Therefore, $G$ is a generalized $KKM^*$ mapping with respect to $F$.

Since $F \in KKM^*(X_\infty,Z)$ and $G$ is a generalized $KKM^*$ mapping with respect to $F$, the family $\{G(x) : x \in X_\infty\}$ has the finite intersection property, and so we conclude that $\cap_{x \in X_\infty} G(x) \neq \phi$. Choose $\eta \in \cap_{x \in X_\infty} G(x)$, then $\eta \in K \setminus (x + \frac{1}{2}U + V)$, for each $x \in X_\infty$. Since $\eta \in \cap_{x \in X_\infty} G(x) \subset K \subset h_{\overline{T(X_\infty)},V}(\overline{T(X_\infty)}) \subset \overline{T(X_\infty)} + V \subset X_\infty + V$, hence there is an $x_0 \in X_\infty$ such that $\eta \in x_0 + \frac{1}{2}U + V$. But $\eta \in K \setminus (x_0 + \frac{1}{2}U + V)$, a contradiction. Therefore, we have proved that for each $U_i \in \mathcal{N}$, there exists an $x_i \in X_\infty$ such that
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\begin{align*}
(x_i + U_i + V) \cap F(x_i) \neq \emptyset. \text{ Let } y_i \in (x_i + U_i + V) \cap F(x_i). \text{ Since } \{y_i\} \subset K \text{ and } K \text{ is compact, we may assume that } \{y_i\} \text{ converges to some } y_0 \in K, \text{ and since } \{x_i\} \subset \overline{X_\infty}, \text{ we assume that } \{x_i\} \text{ converges to } x_0. \text{ The closedness of } F \text{ implies that } (x_0, y_0) \in G_F, \text{ so we have } y_0 \in x_0 + V \text{ and } y_0 \in F(x_0) = h_{T(X_\infty), V}(T(x_0)). \text{ Choose } z_0 \in T(x_0) \text{ such that } y_0 = h_{T(X_\infty), V}(z_0).\text{ Noting that } z_0 \in h_{T(X_\infty), V}(z_0) + V = y_0 + V \subset x_0 + V + V \subset (x_0 + V + V + V), \text{ and hence } T(x_0) \cap x_0 + V + V + V \neq \emptyset \text{ for any } V \in \mathcal{N}, \text{ which just as before, implies } T \text{ has a fixed point.} \quad \square
\end{align*}

**Corollary 3** Let \( X \) be a nonempty bounded convex subset of a locally convex space \( E \), and let \( T \in KKM(X, X) \) is a \( k \)-set contraction, \( 0 < k < 1 \) and closed with \( \overline{T(X)} \subset X \). Then \( T \) has a fixed point in \( X \).

**Corollary 4** Let \( X \) be a nonempty nearly-convex subset of a locally convex space \( E \), and let \( T \in KKM^*(X, X) \) is compact and closed. Then \( T \) has a fixed point in \( X \).

**Corollary 5** [2] Let \( X \) be a nonempty convex subset of a locally convex space \( E \), and let \( T \in KKM(X, X) \) is compact and closed. Then \( T \) has a fixed point in \( X \).

We now establish the following \( KKM^* \)-type theorem, which is equivalent to the matching theorem after it.

**Theorem 5** Let \( X \) be a nonempty bounded nearly-convex subset of a Hausdorff topological vector space \( E \). If \( T, F : X \to 2^X \) are two set-valued mappings satisfying the following:

\begin{enumerate}
  \item \( T \in KKM^*(X, X) \) is a \( k \)-set contraction map, \( 0 < k < 1 \) with \( \overline{T(X)} \subset X \),
  \item for any \( x \in X \), \( F(x) \) is compactly closed in \( X \), and
  \item \( F \) is a generalized \( KKM^* \) mapping with respect to \( T \),
\end{enumerate}

then

\[
\overline{T(X_\infty)} \cap (\cap \{F(x) : x \in X_\infty\}) \neq \emptyset,
\]

where \( X_\infty \) is the precompact-inducing nearly-convex subset of \( X \).
Proof. Let \( \mathcal{N} = \{U_i : i \in I\} \) be a local base of \( E \) such that \( U_i \) is symmetric and open for each \( i \in I \). By the same process of the proof of Theorem 2, we get a compact subset \( T(X_\infty) \) of \( X \), and \( T|_{X_\infty} \in KKM^*(X_\infty, X) \), since \( T \in KKM^*(X, X) \).

Define \( H : X_\infty \to 2^X \) by

\[
H(x) = \overline{T(X_\infty)} \cap F(x), \text{ for each } x \in X_\infty.
\]

It follows from (iii) that \( F \) is a generalized \( KKM^* \) mapping with respect to \( T|_{X_\infty} \), and hence for any \( x \in X_\infty \) and any \( V \in \mathcal{N} \), there exists a continuous convex-inducing \( h_{(x),V} : \{x\} \to X_\infty \) such that \( T(co(h_{(x),V}(\{x\}))) \subset F(x) \) and \( co(h_{(x),V}(x)) \subset X_\infty \). So \( H(x) \neq \emptyset \).

By (ii), \( H(x) \) is compact in \( X \), for each \( x \in X_\infty \). We now claim that \( H \) is a generalized \( KKM^* \) mapping with respect to \( T|_{X_\infty} \). Let \( A \in \langle X_\infty \rangle \).

By (iii), for any \( V \in \mathcal{N} \), there exists a continuous convex-inducing mapping \( h_{A,V} : A \to X_\infty \) such that \( T(co(h_{A,V}(A))) \subset F(A) \) and \( co(h_{A,V}(A)) \subset X_\infty \). So, \( T(co(h_{A,V}(A))) \subset F(A) \cap T(X_\infty) = H(A) \). Thus, we have shown that \( H \) is a generalized \( KKM^* \) mapping with respect to \( T|_{X_\infty} \). Since \( T|_{X_\infty} \in KKM^*(X_\infty, X) \), the family \( \{H(x) : x \in X_\infty\} \) has the finite intersection property. And, since \( H(x) \) is compact, hence \( \bigcap_{x \in X_\infty} H(x) \neq \emptyset \), that is, \( \overline{T(X_\infty)} \cap \bigcap \{F(x) : x \in X_\infty\} \neq \emptyset \). \( \square \)

Corollary 6 Let \( X \) be a nonempty bounded convex subset of a Hausdorff topological vector space \( E \). If \( T, F : X \to 2^X \) are two set-valued mappings satisfying the following:

(i) \( T \in KKM(X, X) \) is a \( k \)-set contraction map, \( 0 < k < 1 \) with \( \overline{T(X)} \subset X \),
(ii) for any \( x \in X \), \( F(x) \) is compactly closed in \( X \), and
(iii) \( F \) is a generalized \( KKM \) mapping with respect to \( T \),

then there exists a nonempty convex subset \( Y \) of \( X \) such that

\[
\overline{T(X_\infty)} \cap \bigcap \{F(x) : x \in X_\infty\} \neq \emptyset,
\]

where \( X_\infty \) is the precompact-inducing nearly-convex subset of \( X \).

As a consequence of the above theorems, we get the following generalization of the Ky Fan matching theorem.

Theorem 6 Let \( X \) be a nonempty bounded nearly-convex subset of a Hausdorff topological vector space \( E \). If \( T, H : X \to 2^X \) are two set-valued mappings satisfying the following:
(i) $T \in KKM^*(X, X)$ is a $k$-set contraction map, $0 < k < 1$ with $\overline{T(X)} \subset X$.

(ii) for any $x \in X$, $H(x)$ is compactly open in $X$, and

(iii) for the precompact-inducing nearly-convex subset $X_\infty$ of $X$, $\overline{T(X_\infty)} \subset H(X_\infty)$,

then the precompact-inducing nearly-convex subset $X_\infty$ of $X$ satisfies the following condition:

$$T(X_\infty) \cap (\cap \{H(x) : x \in M\}) \neq \phi,$$

for some $M \in \langle X_\infty \rangle$.

Proof, Let $N = \{U_i : i \in I\}$ be a local base of $E$ such that $U_i$ is symmetric and open for each $i \in I$. And, by the same process of the proof of Theorem 2, we get a compact subset $\overline{T(X_\infty)}$ of $X$, and $T|_{X_\infty} \in KKM^*(X_\infty, X)$, since $T \in KKM^*(X, X)$.

We claim that there exists $M \in \langle X_\infty \rangle$ such that $T(X_\infty) \cap (\cap \{H(x) : x \in M\}) \neq \phi$. On the contrary, assume that $T(X_\infty) \cap (\cap \{H(x) : x \in M\}) = \phi$ for any $M \in \langle X_\infty \rangle$, then $T(X_\infty) \subset \cap_{x \in M} H^c(x)$. Since $X_\infty$ is nearly-convex, hence for any $V \in \mathcal{N}$, there exists a continuous convex-inducing mapping $h_{M,V} : M \to X_\infty$ such that $\text{co}(h_{M,V}(M)) \subset X_\infty$. So $T(\text{co}(h_{M,V}(M))) \subset T(X_\infty) \subset \cap_{x \in M} H^c(x)$. This implies $H^c$ is a generalized $KKM^*$ mapping with respect to $T$. By (ii), for any $x \in X$, $H^c(x)$ is compactly closed in $X$. Follows Theorem 5, we have $\overline{T(X_\infty)} \cap (\cap \{H^c(x) : x \in X_\infty\}) \neq \phi$, which implies $\overline{T(X_\infty)} \notin \cup_{x \in X_\infty} H(x)$, a contradiction to (iii). We complete the proof. □

Corollary 7 Let $X$ be a nonempty bounded convex subset of a Hausdorff topological vector space $E$. If $T, H : X \to 2^X$ are two set-valued mappings satisfying the following:

(i) $T \in KKM(X, X)$ is a $k$-set contraction map, $0 < k < 1$ with $\overline{T(X)} \subset X$,

(ii) for any $x \in X$, $H(x)$ is compactly open in $X$, and

(iii) for the precompact-inducing convex subset $X_\infty$ of $X$, $\overline{T(X_\infty)} \subset H(X_\infty)$,

then the precompact-inducing convex subset $X_\infty$ of $X$ satisfies the following condition:

$$T(X_\infty) \cap (\cap \{H(x) : x \in M\}) \neq \phi,$$ for some $M \in \langle X_\infty \rangle$.

As a consequence of the above Corollary 6, we have the following generalized variational inequality.

Theorem 7 Let $X$ be a nonempty bounded convex subset of a Hausdorff topological vector space $E$, and let $T \in KKM^*(X, X)$ be a $k$-set contraction map, $0 < k < 1$ with $\overline{T(X)} \subset X$. If $\varphi, \psi : X \times X \to (-\infty, \infty)$ are two real-valued mappings satisfying the following:
(i) \( \psi(x,y) \leq 0 \), for each \((x,y) \in G_T\),
(ii) for fixed \( x \in X \), the mapping \( y \mapsto \varphi(x,y) \) is lower semicontinuous on \( K \) for each compact subset \( K \) of \( X \), and
(iii) for fixed \( y \in X \), the set \( \{ x \in X : \psi(x,y) > 0 \} \) contains the convex hull of the set \( \{ x \in X : \varphi(x,y) > 0 \} \)

then for the precompact-inducing convex subset \( X_\infty \) of \( X \), there exists an \( \overline{y} \in X_\infty \) such that

\[ \varphi(x, \overline{y}) \leq 0 \] for each \( x \in X_\infty \).

**Proof**, Define \( F, S : X \to 2^X \) by

\[ S(x) = \{ y \in X : \psi(x,y) \leq 0 \} \] for each \( x \in X \), and
\[ F(x) = \{ y \in X : \varphi(x,y) \leq 0 \} \] for each \( x \in X \).

By assumption (i), we have \( G_T \subset G_S \), and by assumption (ii), \( F(x) \) is comapctly closed for each \( x \in X \). The condition (iii) implies that for each finite subset \( A \) of \( X \), \( S(co(A)) \subset F(A) \), and then \( T(co(A)) \subset F(A) \), that is; \( F \) is a generalized KKM mapping with respect to \( T \).

So, all the conditions in Corollary 6 are satisfied, and so for the precompact-inducing convex subset \( X_\infty \) of \( X \), we have that \( \overline{T(X_\infty)} \cap (\bigcap \{ F(x) : x \in X_\infty \}) \neq \emptyset \). Let \( \overline{y} \in \overline{T(Y)} \cap (\bigcap \{ F(x) : x \in Y \}) \), and hence we have \( \varphi(x, \overline{y}) \leq 0 \) for each \( x \in Y \). \( \square \)

**References**


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