An Application of the Fibonacci Length

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Abstract

For every integer $n \geq 2$ we define the Fibonacci class of degree 2 of 2-generated groups, and study certain properties of this class. Our method of study is applicable for the groups with more than two generators. This investigation is indeed an application of $FL_A(G)$, the well-known Fibonacci length of the finite group $G = \langle A \rangle$, where $A = \{a_1, a_2, \dots, a_k\}$.

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1. Introduction

Since 1990 the Fibonacci length has been studied and calculated for certain classes of finite groups (one may see [1,2,3,4,5,6] for examples), where the least positive integer l is called the Fibonacci length of the group $G = \langle A \rangle$ with respect to the generating set $A = \{a_1, a_2, \dots, a_n\}$, denoted by $LEN_A(G)$, if it is the period of the sequence

$$x_1 = a_1, x_2 = a_2, \dots, x_n = a_n, \quad x_i = x_{i-n} \cdot x_{i-n+1} \dots x_{i-2} \cdot x_{i-1}, \quad i \geq n+1,$$

of the elements of G . When it is clear which generating set of the group G is being considered we will write $LEN(G)$ for $LEN_A(G)$. Our notation are standard. The Fibonacci sequence of numbers, $\{F_n\}_{-\infty}^{\infty}$, defined by

$$\begin{cases} F_n = F_{n-2} + F_{n-1}, & n \geq 0 \\ F_n = F_{n+2} - F_{n+1}, & \text{otherwise,} \end{cases}$$

where we seed the sequence with $F_0 = 0$ and $F_1 = 1$. Following Wall ([8]), we use the notation $k(n)$ to denote the fundamental period of this sequence. We will also use $C(G)$ and $T(G)$ for the complexity and T-invariant of a finitely presented group G , respectively.

In this paper we give an application of the Fibonacci length on the classification of groups, by using $C(G)$ and $T(G)$. This application is the first theoretical result of this notion after all of its nice numerical results. Following Pervova ([7]) recall two definitions:

Definition 1.1 Let $\langle a_1, a_2, \dots, a_n \mid r_1, r_2, \dots, r_m \rangle$ be a presentation of a group G . The length of this presentation is defined to be $|r_1| + \dots + |r_m|$ where, $|r_i|$ is the length of the word r_i in the alphabet $a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_n^{\pm 1}$. The Complexity is often called the minimum length (of a group). Also $|r_i(a_j)|$ is defined to be the sum of the powers of the generator a_j in the relator r_i , for every i and j .

Definition 1.2 The T -invariant of a finitely presented group G , $T(G)$ is the minimum number t such that G admits a presentation with t relations of length 3 and an arbitrary number of relations of length at most 2.

By the above considerations we now give the following definitions.

Definition 1.3 For a 2-generated non-abelian group G let

$$K_*(G) = \min\{LEN_A(G) \mid G = \langle A \rangle, \quad |A| = 2, \quad LEN_A(G) < \infty\}.$$

Then for every integer $n \geq 2$, the class of all 2-generated groups G satisfying $K_*(G) \mid n$ will be denoted by FL_n , is called the Fibonacci class of length n .

In what follows we investigate the class FL_n for 2-generated groups and our method of investigation will be applicable for the non-abelian groups with more than two generators.

Definition 1.4 For a 2-generated group $G = \langle A \rangle$, where $A = \{a, b\}$, the Co-Fibonacci orbit of G with respect to A , is the sequence

$$y_1 = b, \quad y_2 = a, \quad y_{i+1} = y_{i-1}y_i^{-1}, \quad i \geq 2.$$

By this definition the period of the sequence $\{x_i\}$ is equal to the period of the sequence $\{y_i\}$. Obviously, if $G \in FL_n$ then $y_i = x_{n+3-i}$, for every $i \geq 1$.

2. A classification method

Firstly we consider the following preliminary lemma.

Lemma 2.1 *By the above notations $|x_k| = F_k$, $|y_k| = F_k$, $|x_k(a)| = F_{k-2}$, $|x_k(b)| = F_{k-1}$, $|y_k(a)| = (-1)^k F_{k-1}$ and $|y_k(b)| = (-1)^{k-1} F_{k-2}$.*

Proof. To prove $|x_k| = F_k$ we use an induction method. Indeed, $|x_1| = F_1$ and $|x_2| = F_2$ are true and then $x_k = x_{k-2}x_{k-1}$ gives that $|x_k| = |x_{k-2}| + |x_{k-1}| = F_k$. Other statements may be proved in a similar way. $\square$

The following proposition is of interest to consider and one may see the proof in Pervova ([7]).

Proposition 2.2 *For a finitely presented group G ,*

(i) $T(G) = \min\{\sum_{i=1}^m \max\{|r_i|, -2, 0\} \mid G = \langle a_1, a_2, \dots, a_n \mid r_1, r_2, \dots, r_m \rangle\}$.

(ii) $C(G) \geq \log_2^{|Tor(\bar{G}/\bar{G}')|}$, where $Tor(H)$ is used for the torsion part of an abelian group H .

Proposition 2.3 *For a finitely presented group $G = \langle X \mid R \rangle$, $G \in FL_n$ if and only if $G = \langle a, b \mid R \rangle$ where*

$$\{x_{[n/2]+1} \cdot y_{[(n+1)/2]+2}^{-1}, x_{[n/2]+2} \cdot y_{[(n+1)/2]+1}^{-1}\} \subseteq R.$$

Proof. Let $G \in FL_n$. Then there exists a generating set $A = \langle a, b \rangle$ for G such that $K_*(G) = LEN_A(G)$ and $K_*(G) \mid n$. By using the relations $x_{n+1} = x_{n-1}x_n$ and $y_{n+1} = y_{n-1}y_n^{-1}$ we get

$$\begin{cases} x_{n+1} = a = y_2 \\ x_{n+2} = b = y_1, \end{cases}$$

and then we get $x_n = ba^{-1} = y_3$. Continuing this process yields

$$\{x_{[n/2]+1} \cdot y_{[(n+1)/2]+2}^{-1}, x_{[n/2]+2} \cdot y_{[(n+1)/2]+1}^{-1}\} \subseteq R.$$

Conversely, let $G = \langle X \mid R \rangle$ and the above relation holds, then we get

$$\begin{cases} x_{[n/2]+1} = y_{[(n+1)/2]+2} \\ x_{[n/2]+2} = y_{[(n+1)/2]+1}, \end{cases}$$

or,

$$\begin{cases} x_{[n/2]+1} \cdot x_{[n/2]+2} = y_{[(n+1)/2]+2} \cdot y_{[n/2]+1} \\ x_{[n/2]+2} = y_{[(n+1)/2]+1}. \end{cases}$$

Consequently, $x_{[n/2]+3} = y_{[(n+1)/2]}$ and $x_{[n/2]+2} = y_{[(n+1)/2]+1}$. Finally, this process yields $x_{n+2} = y_1 = b$ and $x_{n+1} = y_2 = a$. So, $G \in FL_n$. $\square$

As a result of this proposition and using the well-known von-Dyke's theorem (on the presentation of groups) we get:

Corollary 2.4 For a finitely presented group G , $G \in FL_n$ if and only if G is a homomorphic image of the group $\langle a, b \mid r_1, r_2 \rangle$ where,

$$r_1 = x_{[n/2]+1} \cdot y_{[(n+1)/2]+2}^{-1}, \quad r_2 = x_{[n/2]+2} \cdot y_{[(n+1)/2]+1}^{-1}.$$

Proposition 2.5 Let $G = \langle a, b \mid r_1 = x_{[n/2]+1} \cdot y_{[(n+1)/2]+2}^{-1}, r_2 = x_{[n/2]+2} \cdot y_{[(n+1)/2]+1}^{-1} \rangle$. Then,

- (i) for even values of n , $T(G) \leq 2F_{(n/2)+3} - 4$,
- (ii) for odd values of n , $T(G) \leq F_{(n+9)/2} - 4$.

Proof. For even values of n we get $|r_1| = |r_2| = F_{(n/2)+1} + F_{(n/2)} + 3 = F_{(n/2)+3}$. Then (i) follows by using the Proposition 2. A similar method for odd values of n shows that $|r_1| = F_{(n+1)/2} + F_{(n+5)/2}$ and $|r_2| = 2F_{(n+3)/2}$. Therefore, $T(G) \leq F_{(n+5)/2} + F_{(n+7)/2} - 4 = F_{(n+9)/2} - 4$. $\square$

Corollary 2.6

(i) A finitely presented group G belongs to the class FL_3 if and only if $G = \langle a, b \mid R \rangle$ where, $\{abab^{-1}, baba^{-1}\} \subseteq R$. In particular, $Q_8 \in FL_3$.

(ii) A finitely presented group G belongs to the class FL_6 if and only if $G = \langle a, b \mid R \rangle$ where, $\{a^2ba^2b^{-1}, b^2ab^2a^{-1}\} \subseteq R$. In particular, $D_{2n}, D_\infty \in FL_3$.

Proof. By the Corollary 2.4, $G \in FL_3$ if and only if $\{x_2y_4^{-1}, x_3y_3^{-1}\} \subseteq R$. And obviously this is equivalent to $\{b(a^2b^{-1})^{-1}, ab(ba^{-1})^{-1}\} \subseteq R$. Now, (i) follows by a simple manipulation of the relations. The assertion (ii) may be proved in a similar way and $G \in FL_6$ is equivalent to $\{x_4y_5^{-1}, x_5y_4^{-1}\} \subseteq R$, i.e.; $\{bab(ba^{-1}ba^{-2})^{-1}, ab^2ab(a^2b^{-1})^{-1}\} \subseteq R$. The result is now immediate.

For the particular cases consider the presentations $Q_8 = \langle a, b \mid a^4, a^2b^{-2}, b^{-1}aba \rangle$, $D_{2n} = \langle a, b \mid a^2, b^2, (ab)^n \rangle$ and $D_\infty = \langle a, b \mid a^2, b^2 \rangle$. $\square$

3. Conclusion

The relation matrix of the quotient groups G/G' , for every groups $G \in FL_n$ depends on the Fibonacci numbers. Indeed, we have:

Proposition 3.1 Let $G = \langle a, b \mid x_{[n/2]+1} = y_{[(n+1)/2]+2}, x_{[n/2]+2} = y_{[(n+1)/2]+1} \rangle$.

Then $M(G/G')$, the relation matrix of the group G/G' is equal to:

$$\begin{aligned} M_1 &= \begin{pmatrix} F_{n/2} & 0 \\ 0 & 5F_{n/2} \end{pmatrix}, & \text{if } n \equiv 0 \pmod{4}, \\ M_2 &= \begin{pmatrix} F_{n/2-1} + F_{n/2+1} & 0 \\ 0 & F_{n/2-1} + F_{n/2+1} \end{pmatrix}, & \text{if } n \equiv 2 \pmod{4}, \\ M_3 &= \begin{pmatrix} F_{(n-3)/2} + F_{(n+3)/2} & -F_{(n-3)/2} \\ -F_{(n-3)/2} & -F_{(n+3)/2} \end{pmatrix}, & \text{if } n \equiv 1 \pmod{4}, \\ M_4 &= \begin{pmatrix} -2F_{(n-1)/2} & F_{(n+3)/2} \\ F_{(n+3)/2} & -F_{(n-3)/2} \end{pmatrix}, & \text{if } n \equiv 3 \pmod{4}. \end{aligned}$$

Proof. We consider two cases for n and use the Lemma 2.1. If n is even then

$$\begin{aligned} |r_1(a)| &= F_{n/2-1} + (-1)^{n/2+1} F_{n/2+1}, & |r_1(b)| &= F_{n/2} + (-1)^{n/2} F_{n/2}, \\ |r_2(a)| &= F_{n/2} + (-1)^{n/2} F_{n/2}, & |r_2(b)| &= F_{n/2+1} + (-1)^{n/2+1} F_{n/2-1}. \end{aligned}$$

So the result follows in two cases when $n \equiv 0$ or $2 \pmod{4}$.

For the odd values of n we get

$$\begin{aligned} |r_1(a)| &= F_{(n-3)/2} + (-1)^{(n-1)/2} F_{(n+3)/2}, & |r_1(b)| &= F_{(n-1)/2} + (-1)^{(n+1)/2} F_{(n+1)/2}, \\ |r_2(a)| &= F_{(n-1)/2} + (-1)^{(n-3)/2} F_{(n+1)/2}, & |r_2(b)| &= F_{(n+1)/2} + (-1)^{(n-1)/2} F_{(n-1)/2}. \end{aligned}$$

If $n \equiv 3 \pmod{4}$, then

$$M(G/G') = \begin{pmatrix} F_{(n-3)/2} - F_{(n+3)/2} & F_{(n-1)/2} + F_{(n+1)/2} \\ F_{(n-1)/2} + F_{(n+1)/2} & F_{(n+1)/2} - F_{(n-1)/2} \end{pmatrix}$$

which is equivalent to the matrix M_4 . And if $n \equiv 1 \pmod{4}$, then we get

$$M(G/G') = \begin{pmatrix} F_{(n-3)/2} + F_{(n+3)/2} & F_{(n-1)/2} - F_{(n+1)/2} \\ F_{(n-1)/2} - F_{(n+1)/2} & F_{(n-1)/2} + F_{(n+1)/2} \end{pmatrix}$$

which is equivalent to the matrix M_3 . This completes the proof. $\square$

As an example we may consider the group $H = \langle a, b \mid a^2ba^2 = b, b^2ab^2 = a \rangle$ and by the propositions 2.2 and 3.1 we get $|H/H'| = (F_2 + F_4)^2 = 16$ and $C(H) \geq 4$.

Corollary 3.2 For every even values of n if $G \in FL_n$ then $|G/G'|$ divides $5F_{n/2}^2$ or $(F_{n/2-1} + F_{n/2+1})^2$ if $n \equiv 0 \pmod{4}$ or $n \equiv 2 \pmod{4}$, respectively.

Proof. By the Corollary 2.4 and the Proposition 3.1. $\square$

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