New ROW-type Scheme with Weak Order 2 for Approximating Stochastic Differential Equations

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Abstract

In this paper, we will define a new Rosenbrock-Wanner type scheme (ROW) for solving stochastic differential equations (SDEs). By applying the corresponding order conditions, an implicit four stage ROW-type scheme with weak order 2 be introduced and compared its results with the explicit ten stage ROW-type scheme.

Keywords: Stochastic differential equations, ROW method, order conditions, weak order

1 Introduction

Until recently, many models have developed to describe physical phenomena have ignored stochastic effects because of the lack of suitable numerical methods. Stochastic differential system are made by adding stochastic effects to ordinary differential equations.

Some areas where SDEs are used in modeling include investment finance, population dynamics, polymer dynamics and VLSI circuit design. There are also stochastic counterparts of, for example, chemical kinetic models.

One of suitable methods for solving SDEs, is ROW-type scheme, where it has higher convergence order with respect to the Euler-Maruyama and Heun methods[2]. The aim of this paper is presentation of an implicit four stage ROW-type scheme and comparison with explicit ten stage ROW, defined in[6].

Assume $W(t)$ be a Wiener process define on the probability space $(\Omega, \mathcal{F}, P)$. A SDE given as follows:

$$dY(t) = f(Y(t))dt + g(Y(t)) \circ dW(t), \quad Y(t_0) = y_0 \quad (1.1)$$

Where $f(Y(t))$ and $g(Y(t))$ are drift and diffusion coefficients respectively, and the symbol $\circ$ present Stratonovich integral.

Definition 1.1. Let $\bar{Y}_N$ be numerical approximation of $Y(t_N)$ after N steps with constant step size $h = \frac{t_N - t_0}{N}$; then $\bar{Y}$ converge strongly to Y with order p if $\exists k > 0$ (independent of h) and $\delta > 0$ such that $E(|\bar{Y}_N - Y(t_N)|) \leq kh^p$, $h \in (0, \delta)$.

Definition 1.2. The discrete time approximation Y_N converge weakly with order p to Y if for each polynomial g (which is $2(p+1)$ time continuously differentiable), $\exists k > 0$ (independent of h) and $\delta > 0$ such that

$$|E(g(Y_N)) - E(g(Y(t_N)))| \leq kh^p, \quad h \in (0, \delta).$$

If we assume $g(y) = ay + b$, then

$$|E(Y(t_N)) - E(Y_N)| \leq kh^p. \quad (1.2)$$

By using the following approximation, the value of k can be estimated

$$k \approx \frac{1}{h^p} [|E(Y(t_N)) - E(Y_N)|] \quad (1.3)$$

1.1 Richardson extrapolation method

Richardson method is one of estimating method for the order of numerical methods. Suppose order of a numerical method for solving ordinary differential equation (ODEs) be p , and h_1, h_2, h_3 be different time steps on interval $[0, T]$. If t_0 be a fixed point in $[0, T]$ and let y_1, y_2, y_3 approximate ODE solution at point t_0 for different step sizes h_1, h_2, h_3 respectively, then

$$Y(t_0) - y_1 \simeq kh_1^{p+1}, \quad Y(t_0) - y_2 \simeq kh_2^{p+1}, \quad Y(t_0) - y_3 \simeq kh_3^{p+1},$$

therefore

$$\frac{Y(t_0) - y_1}{Y(t_0) - y_2} \simeq \left(\frac{h_1}{h_2}\right)^{p+1} \quad \frac{Y(t_0) - y_2}{Y(t_0) - y_3} \simeq \left(\frac{h_2}{h_3}\right)^{p+1}.$$

$Y(t_0)$ can be obtained by two following formulae:

$$Y(t_0) \simeq \frac{y_1 - y_2 \left(\frac{h_1}{h_2}\right)^{p+1}}{1 - \left(\frac{h_1}{h_2}\right)^{p+1}} \quad Y(t_0) \simeq \frac{y_2 - y_3 \left(\frac{h_2}{h_3}\right)^{p+1}}{1 - \left(\frac{h_2}{h_3}\right)^{p+1}}.$$

Now, assume $h_1 = 2h_2 = 4h_3$, then

$$\begin{aligned} y_1 - 2^{p+1}y_2 &\simeq y_2 - 2^{p+1}y_3 \Rightarrow 2^{p+1} \simeq \frac{y_2 - y_1}{y_3 - y_2} \\ \Rightarrow p &\simeq -1 + \frac{1}{\log 2} \log \frac{y_2 - y_1}{y_3 - y_2} \end{aligned}$$

We apply this method for SDEs and take into consideration of definition 1.2, with linear weak convergency

$$p \simeq \frac{1}{\log 2} \log \frac{E(Y(h_2)) - E(Y(h_1))}{E(Y(h_3)) - E(Y(h_2))}$$

2 ROW method

The s -stage ROW-type scheme is as follows:

$$\begin{aligned} y_h &= y_0 + \sum_{j=1}^s c_j Y_j \\ Y_i &= h_i a_i f(y_0 + \sum_{j=1}^s \alpha_{ij} Y_j) + h_i f^{(1)}(y_0) \sum_{j=1}^s \gamma_{ij} Y_j \\ &\quad + w_i b_i g(y_0 + \sum_{j=1}^s \beta_{ij} Y_j) + w_i g^{(1)}(y_0) \sum_{j=1}^s \delta_{ij} Y_j \quad (i = 1, \dots, s) \end{aligned} \quad (2.1)$$

Where y_h is approximation of the solution (1.1) and

$$\gamma_{ij} = 0 \ (i < j), \quad h_i = h, \quad \alpha_{ij} = \beta_{ij} = \gamma_{ij} = 0 \ (i \leq j).$$

With the definition of $J_{j_1 j_2 \dots j_k}$ as J - integral of Stratonovich multiple integral, where integration is with respect to ds , if $j_i = 0$ and $\circ dW(s)$, if $j_i = 1$. Stratonovich Taylor expansion of equation (1.1) is obtained as following:

$$\begin{aligned} y_h &= y_0 + f(y_0)J_0 + g(y_0)J_1 \\ &\quad + f'(y_0)(f(y_0))J_{00} + f'(y_0)(g(y_0))J_{10} \\ &\quad + g'(y_0)(f(y_0))J_{01} + g'(y_0)(g(y_0))J_{11} + \dots \end{aligned} \quad (2.2)$$

In equation (2.1), by writing Taylor expansions of $f(y_0 + \sum_{j=1}^s \alpha_{ij} Y_j)$ and $g(y_0 + \sum_{j=1}^s \beta_{ij} Y_j)$ at y_0 and comparing with (2.2) of method with order two, following system be obtained, where include 17 equations and $s(2s + 3)$ unknown variables[5].

$$\begin{aligned} 1) \quad &\sum_{i=1}^s c_i w_i b_i = W(h) \\ 2) \quad &\sum_{i=1}^s c_i h_i a_i = h \\ 3) \quad &\sum_{i,j=1}^s c_i w_i (b_i \beta_{ij} + \delta_{ij}) w_j b_j = \frac{1}{2} W^2(h) \\ 4) \quad &\sum_{i,j=1}^s c_i w_i (b_i \beta_{ij} + \delta_{ij}) h_j a_j = \int_0^h s \circ dW(s) \end{aligned}$$

- 5) $\sum_{i,j=1}^s c_i h_i (a_i \alpha_{ij} + \gamma_{ij}) w_j b_j = \int_0^h W(s) ds$
- 6) $\sum_{i,j=1}^s c_i h_i (a_i \alpha_{ij} + \gamma_{ij}) h_j a_j = \frac{1}{2} h^2$
- 7) $\sum_{i,j,l=1}^s c_i w_i b_i \beta_{ij} w_j b_j \beta_{il} w_l b_l = \frac{1}{3} W^3(h)$
- 8) $\sum_{i,j,k=1}^s c_i w_i b_i \beta_{ij} w_j b_j \beta_{ik} h_k a_k = \int_0^h s W(s) \circ dW(s)$
- 9) $\sum_{i,j,k=1}^s c_i h_i a_i \alpha_{ij} w_j b_j \alpha_{ik} w_k b_k = \int_0^h W^2(s) ds$
- 10) $\sum_{i,j,k=1}^s c_i w_i (b_i \beta_{ij} + \delta_{ij}) w_j (b_j \beta_{jk} + \delta_{jk}) w_k b_k = \frac{1}{6} W^3(h)$
- 11) $\sum_{i,j,k=1}^s c_i w_i (b_i \beta_{ij} + \delta_{ij}) h_j (a_j \alpha_{jk} + \delta_{jk}) w_k b_k = \int_0^h \int_0^s W(u) du \circ dW(s)$
- 12) $\sum_{i,j,k=1}^s c_i w_i (b_i \beta_{ij} + \delta_{ij}) w_j (b_j \beta_{jk} + \delta_{jk}) h_k a_k = \int_0^h \int_0^s u \circ dW(u) \circ dW(s)$
- 13) $\sum_{i,j,k=1}^s c_i h_i (a_i \alpha_{ij} + \gamma_{ij}) w_j (b_j \beta_{jk} + \delta_{jk}) w_k b_k = \int_0^h W^2(s) ds$
- 14) $\sum_{i,j,k,l=1}^s c_i w_i b_i \beta_{ij} w_j b_j \beta_{ik} w_k b_k \beta_{il} w_l b_l = \frac{1}{4} W^4(h)$
- 15) $\sum_{i,j,k,l=1}^s c_i w_i b_i \beta_{ij} w_j b_j \beta_{ik} w_k (b_k \beta_{kl} + \delta_{kl}) w_l b_l = \frac{1}{8} W^4(h)$
- 16) $\sum_{i,j,k,l=1}^s c_i w_i (b_i \beta_{ij} + \delta_{ij}) w_j (b_j \beta_{jk} + \delta_{jk}) w_k (b_k \beta_{kl} + \delta_{kl}) w_l b_l = \frac{1}{24} W^4(h)$
- 17) $\sum_{i,j,k,l=1}^s c_i w_i (b_i \beta_{ij} + \delta_{ij}) w_j b_j \beta_{jk} w_k b_k \beta_{jl} w_l b_l = \frac{1}{12} W^4(h)$

The above order conditions are obtained in the following way: We are looking for a ROW method with order two, so elementary weights of all bi-colored rooted trees should be equal to elementary numerical weights of all bi-colored rooted trees with labels which satisfy in $r(t) + r_b(t) \leq 4$ ($r(t)$ and $r_b(t)$ are the number of bullet vertices and the number of all vertices in tree t , respectively). **Definition 2.1.** An elementary weight of bi-colored rooted tree t is given recursively, as follow:

$$\Phi(\tau) = I[1] \quad , \quad \Phi(\sigma) = J[1]$$

$$\Phi(t) = \begin{cases} I[\prod_{i=1}^k \Phi(t_i)] & t = [t_1, \dots, t_k] \\ J[\prod_{i=1}^k \Phi(t_i)] & t = \{t_1, \dots, t_k\}, \end{cases}$$

where

$$I[H](s) = \int_0^s H(y(s_1)) ds_1 \quad J[H](s) = \int_0^s H(y(s_1)) \circ dW(s_1)$$

Definition 2.2. An elementary numerical weight as $(s+1)$ -th component of vector $\bar{\Phi}(t)$ for a bi-colored rooted tree with labels t be defined recursively:

$$\begin{aligned} \bar{\Phi}(\tau_X) &= [1, \dots, 1] D_a X, & \bar{\Phi}(\sigma_X) &= [1, \dots, 1] D_b X \\ \bar{\Phi}(t) &= \begin{cases} (\prod_{i=1}^k \bar{\Phi}(t_i)) D_a X & t = [t_1, \dots, t_k]_X \\ (\prod_{i=1}^k \bar{\Phi}(t_i)) D_b X & t = \{t_1, \dots, t_k\}_X, \end{cases} \end{aligned}$$

where X is the label of vertices and would be one of matrices A, B, A', B' ,

$$\begin{aligned} A &= \begin{pmatrix} \alpha_{11} & \dots & \alpha_{s+1,1} \\ \vdots & & \vdots \\ \alpha_{1s} & \dots & \alpha_{s+1,s} \\ 0 & \dots & 0 \end{pmatrix} & A' &= \begin{pmatrix} \alpha_{11} + \bar{\gamma}_{11} & \dots & \alpha_{s+1,1} + \bar{\gamma}_{s+1,1} \\ \vdots & & \vdots \\ \alpha_{1s} + \bar{\gamma}_{1s} & \dots & \alpha_{s+1,s} + \bar{\gamma}_{s+1,1} \\ 0 & \dots & 0 \end{pmatrix} \\ B &= \begin{pmatrix} \beta_{11} & \dots & \beta_{s+1,1} \\ \vdots & & \vdots \\ \beta_{1s} & \dots & \beta_{s+1,s} \\ 0 & \dots & 0 \end{pmatrix} & A' &= \begin{pmatrix} \beta_{11} + \bar{\delta}_{11} & \dots & \beta_{s+1,1} + \bar{\delta}_{s+1,1} \\ \vdots & & \vdots \\ \beta_{1s} + \bar{\delta}_{1s} & \dots & \beta_{s+1,s} + \bar{\delta}_{s+1,1} \\ 0 & \dots & 0 \end{pmatrix} \\ D_a &= \begin{pmatrix} h_1 a_1 & & 0 \\ & \ddots & \\ 0 & & h_{s+1} a_{s+1} \end{pmatrix} & D_b &= \begin{pmatrix} w_1 b_1 & & 0 \\ & \ddots & \\ 0 & & w_{s+1} b_{s+1} \end{pmatrix} \end{aligned}$$

For example the elementary weight of $t = \sigma$ ($\circ$) and the elementary numerical weight of σ_A ($\circ_A$) are:

$$\begin{aligned} \Phi(\sigma) &= J[1] = \int_0^h \circ dW(s) = W(h) \\ \bar{\Phi}(\sigma_A) &= [1, \dots, 1] D_b A = \left(\sum_{i=1}^s w_i b_i \alpha_{1i}, \dots, \sum_{i=1}^s w_i b_i \alpha_{s+1,i} \right) \end{aligned}$$

Therefore elementary numerical weight is the $(s+1)$ -st component of $\bar{\Phi}(\sigma_A)$ and according to the definition $\alpha_{s+1,i} = c_i$ will be equal $\sum_{i=1}^s w_i b_i \alpha_{s+1,i}$. So that $\sum_{i=1}^s w_i b_i c_i = W(h)$, and this is the first equation of the order conditions.

Lemma 2.3. There is no three stages ROW method with weakly order 2.

Proof. Consider a three stage ROW method with weakly order $p = 2$. So two matrices β and δ of the method have the follows structure:

$$\delta = \begin{pmatrix} 0 & 0 & 0 \\ \delta_{21} & 0 & 0 \\ \delta_{31} & \delta_{32} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & 0 & 0 \\ \beta_{21} & 0 & 0 \\ \beta_{31} & \beta_{32} & 0 \end{pmatrix}$$

parameters	specifications
α_{ij}, γ_{ij}	$\alpha_{21} = \gamma_{21} = 0.5, \alpha_{31} = \gamma_{31} = 1, \alpha_{32} = \gamma_{32} = 0.5, \alpha_{41} = 1, \gamma_{41} = -0.3611$ $, \alpha_{42} = \gamma_{42} = 1, \alpha_{43} = \gamma_{43} = 0.5, \gamma_{33} = 1, \text{ otherwise } 0$
β_{ij}, δ_{ij}	$\beta_{21} = \delta_{21} = -3, \beta_{31} = \delta_{31} = -21, \beta_{32} = \delta_{32} = -4.5, \beta_{41} = 1, \delta_{41} = -1$ $, \beta_{42} = \delta_{42} = -0.5166, \beta_{43} = \delta_{43} = 0.0166, \text{ otherwise } 0$
a_i, b_i	$a_1 = b_1 = -0.5, a_4 = b_4 = -1.0925, \text{ otherwise } 1$
c_i	$c_1 = -3.8425, c_2 = 0.1728, c_3 = -0.0015, c_4 = 1$
w_i	$w_1 = w_2 = w_3 = w_4 = \Delta W$

Table 1: Parameters of the implicit four stage ROW method.

Now by applying the 16-th equation of the order conditions, we have:

1. if $i = 1$ then $\beta_{1j} = \delta_{1j} = 0$,
2. if $i = 2$ So that $\delta_{2j} \neq 0$ and $\beta_{2j} \neq 0$, then j must be equal one, and so $\beta_{1k} = \delta_{1k} = 0$,
3. For $i = 3$, the only new case is $j = 2$, therefore $k = 1$, and so $\beta_{1l} = \delta_{1l} = 0$,

In all of the above cases, left hand side of the 16-th equation of the order conditions will be equal zero and this is contradiction. Consequently the minimum stage number for a ROW method with $p = 2$ is four.

2.1 Implicit four stage ROW method

Suppose $s = 4$ in (2.2). In all of the order equations of the pervious section, except the equations 4) and 5) which their right-hand sides are $\frac{1}{2}h.\Delta W$, the integral in right-hand side of the equations 8), 9), 11), 12) and 13) can't be calculated. By evaluation $w_1, w_2, w_3, w_4, a_1, a_2, a_3, b_1, b_2, b_3, \alpha_{21}, \alpha_{32}, \alpha_{43}, \alpha_{42}, \alpha_{31}, \gamma_{33}, c_4$, as in Table 1, also with conditions $\alpha_{ij} = \gamma_{ij}, \beta_{ij} = \delta_{ij}$ for all i, j except $\gamma_{33} \neq 0$ and $\alpha_{41} \neq \gamma_{41}$, a system of 12 nonlinear equations with 25 unknown are obtained. We have solved this system in MAPLE environment. So the implicit four stage ROW method be defined such as Table 1 ($\gamma_{33} \neq 0$). Explicit four stage ROW methods were calculated from the above system with different initial evaluation, but they hadn't good accuracy in comparison with the implicit method.

2.2 Explicit ten stage ROW method

Komori et. al.[5] assumed $s = 10$ in (2.2) and derived an explicit ten stage ROW method such as Table 2.

Figure 1: Comparison of exact and numerical solutions of the population dynamics SDE with two ROW methods .

parameters	specifications
$a_i, \alpha_{ij}, \gamma_{ij}, \delta_{ij}$	$a_3 = a_5 = 1, \alpha_{51} = \alpha_{53} = 1, \gamma_{32} = \frac{1}{2}$, otherwise 0
b_i	$b_3 = 0, b_5 = -\frac{1}{12}$, otherwise 1
c_i	$c_1 = c_3 = c_5 = \frac{1}{2}, c_6 = \frac{1}{24}, c_7 = \frac{1}{3}, c_8 = -\frac{1}{3},$ $c_9 = c_{10} = \frac{1}{4}$, otherwise 0
w_i	$w_1 = \Delta W, w_2 = w_3 = w_4 = w_5 = w_6 = w_7 = w_8 = \sqrt{h}$ $w_9 = \Delta W + ((\Delta W)^2 - h)h^{-\frac{1}{2}}$ $w_{10} = \Delta W - ((\Delta W)^2 - h)h^{-\frac{1}{2}}$
β_{ij}	$\beta_{41} = \beta_{43} = \beta_{51} = \beta_{53} = \beta_{54} = \beta_{61} = \beta_{63} = \beta_{71} = \beta_{73} =$ $\beta_{81} = \beta_{83} = \beta_{92} = \beta_{93} = \beta_{10,3} = 1, \beta_{64} = -1, \beta_{74} = \frac{1}{2},$ $\beta_{84} = -\frac{1}{2}, \beta_{10,2} = -1$, otherwise 0

Table 2: Parameters of the explicit ten stage ROW method.

3 Numerical results

One applications of SDEs is the population dynamics([1,4]). The simplest deterministic model of the population growth is a exponential equation $dy(t) = ay(t)dt$, where a is the Malthusian growth coefficient, which is usually a positive constant. The variation of environmental conditions may change the coefficient a to $a + \sigma\xi_t$, where ξ_t is a white noise process.

The population dynamics be modeled by the following stochastic differential equation:

$$dY(t) = (\lambda Y(t) - Y^2(t))dt + \sigma Y(t)dW_t,$$

Where has the general exact solution([3]):

$$Y_t = \frac{y_0 \exp((\lambda - \frac{1}{2}\sigma^2)t + \sigma W_t)}{1 + y_0 \int_0^t \exp((\lambda - \frac{1}{2}\sigma^2)s + \sigma W_s)ds}.$$

In this section, the numerical results of two ROW-type schemes are compared and demonstrated the advantages of using the implicit four stage ROW method. All numerical simulations with two methods have obtained with similar codes in Matlab.

Figure 1, shows the exact and numerical simulation of the stochastic population dynamics with these two methods. For numerical simulation, we consider $\lambda = 0.01, \sigma = 1, y_0 = 1$ and $h = 2^{-9}$ and random numbers with the standard normal distribution in time interval $[0, 1]$ be generated in Matlab with the initial seed equal 11.

Figure 2, plots the CPU time of the implicit four stage and the explicit ten stage ROW methods for different step sizes $h = 2^{-4}, 2^{-5}, \dots, 2^{-10}$. This figure shows the computational cost of the implicit four stage is less than the explicit ten stage method.

Now we would like to estimate the upper bound kh^p in (1.2). By applying Richardson extrapolation method and numerical approximation of the population dynamic equation for $h_1 = 2^{-6}, h_2 = 2^{-7}$ and $h_3 = 2^{-8}$, order of the implicit four stage and the explicit ten stage ROW methods are estimated 2.0799 and 1.9742, respectively.

In Table 3, the constant k be estimated by using (1.3) for $h = 2^{-6}, 2^{-7}$ and 2^{-8} and finally Table 4 gives the estimation value of kh^p in the right-hand side of (1.2). All numerical results are obtained by 1000 simulated trajectories.

h	2^{-6}	2^{-7}	2^{-8}
implicit four stage	275	1163	4916
explicit ten stage	256	1005	3949

Table 3: Estimation of the coefficient k , for two ROW methods.

Figure 2: CPU time of numerical approximation of the population dynamics SDE with two ROW-type schemes versus h .

Our numerical experiment shows that the implicit four stage ROW method has less CPU time, higher estimation of order(2.0799) and lower upper bound kh^p in Table 4 with respect to the explicit ten stage ROW method.

h	2^{-6}	2^{-7}	2^{-8}
implicit four stage	0.04815	0.04817	0.04816
explicit ten stage	0.06957	0.06952	0.06952

Table 4: Estimation of the value of kh^p , for two ROW methods.

By adding the stage number to a ROW method, we may define new methods with higher accuracy and certainly with much computational expensive. By Lemma 2.3, $s = 4$ is the minimum stage number can be supposed for a ROW method with order $p = 2$.

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Received: October 25, 2006