Some Strong Deviation Theorems
for Dependent Stochastic Sequence

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Abstract. The upper and lower bounds of the sequence of sums
\[ \frac{1}{a_n} \sum_{k=1}^{n} \left\lfloor \frac{X_k - \mu_{k,n}}{b_n} \right\rfloor \]
are obtained, where \( \{b_n, n \in \mathbb{N}\} \) are regularly vary functions
and \( \{a_n, n \in \mathbb{N}\} \) are real constants satisfying \( a_n \to \infty \) as \( n \to \infty \). Some strong
deviation theorems for arbitrary continuous random sequence \( \{X_n, n \in \mathbb{N}\} \) are
obtained under suitable conditions.

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1. Introduction

Let \((\Omega, \mathcal{F}, P)\) be a probability space, and let \( \{\mathcal{F}_n, n \in \mathbb{N}\} \) be an increasing
sequence of sub-\(\sigma\)-fields of \( \mathcal{F} \), and suppose that \( \{X_n, \mathcal{F}_n, n \in \mathbb{N}\} \) be a sto-
chastic sequence defined on this probability space, with the joint distribution
densities
\[ p_n(x_1, \ldots, x_n) > 0, \quad n \in \mathbb{N} \quad (1.1) \]
and \( p_k(x), k \in \mathbb{N} \) be the marginal density functions of them, let
\[ \Pi = \pi_n(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n} p_k(x_k), \quad n \in \mathbb{N}. \quad (1.2) \]

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Definition 1. Let \( \{X_n, n \in \mathbb{N}\} \) be a sequence of random variables. Furthermore, suppose that \( \{a_n, n \in \mathbb{N}\} \) be a sequence of positive reals increasing to \( \uparrow \infty \), and set

\[
T_n(\omega) = \frac{\pi_n(X_1, \ldots, X_n)}{p_n(X_1, \ldots, X_n)}, \quad \gamma_n(\omega) = \log T_n(\omega),
\]

The random variable

\[
\gamma(\omega) = -\lim inf_n \frac{\gamma_n(\omega)}{a_n}
\]

is called the limit logarithmic likelihood ratio, relative to the product of one-dimensional marginals of \( \{X_n, n \in \mathbb{N}\} \), where \( \log \) is the natural logarithm, \( \omega \) is the sample point, and \( X_k \) stands for \( X_k(\omega) \).

Definition 2. Let \( \{X_n, n \in \mathbb{N}\} \) be a sequence of nonnegative random variables, and is said to be:

1) stochastically dominated by a nonnegative random variable \( X \) (we write \( \{X_n, n \in \mathbb{N}\} \prec X \)) if there exists a constant \( C > 0 \) such that

\[
\sup_{n \geq 1} P\{X_n > x\} \leq CP\{X > x\} \quad \text{for all } x > 0.
\]

2) stochastically dominated in Cesàro sense by a nonnegative random variable \( X \) (we write \( \{X_n, n \in \mathbb{N}\} \prec X(C) \)) if there exists a constant \( C > 0 \) such that

\[
\sup_{n \geq 1} D_n^{-1} \sum_{k=1}^{n} d_k P\{X_k > x\} \leq CP\{X > x\} \quad \text{for all } x > 0.
\]

where \( \sum_{k=1}^{n} d_k = \mathcal{O}(D_n) \).

Definition 3. Let \( a > 0 \). A positive measurable function \( u \) on \([a, \infty)\) varies regularly at infinity with exponent \( \rho, -\infty < \rho < \infty \), denoted \( u \in \mathcal{RV}(\rho) \), if and only if

\[
\frac{u(tx)}{u(t)} \to \infty \quad \text{as } t \to \infty \quad \text{forall } x > 0.
\]

If \( \rho = 0 \) the function is slowly varying at infinity; \( u \in \mathcal{SV} \).

The basis for proving the Strong Deviation Theorems is the "convergence of likelihood ratio" which we, however, quote because of its central role in this paper.

Lemma 1. (See e.g. Chung K.L. 1988) Let \( p_n(x_1, \ldots, x_n), h_n(x_1, \ldots, x_n) \) be two probability density functions on \( \{\Omega, \mathcal{F}, P\} \) let

\[
\Lambda_n(\omega) = \frac{h_n(X_1, \ldots, X_n)}{p_n(X_1, \ldots, X_n)}.
\]
then
\[
\limsup a_n^{-1} \log \Lambda_n(\omega) \leq 0, \quad a.s. \tag{1.7}
\]

2. Main Results and Proofs

**Theorem 1.** Let \( \{X_n, n \in \mathbb{N}\} \) be a sequence of random variables with \( \{X_n, n \in \mathbb{N}\} \prec X \) and \( \gamma(\omega) \) be defined as above. Further, let, for \( x > 0, b \in \mathcal{R} \mathcal{V}(1/\rho), \) for some \( \rho \in (0, 1], \) that is, \( b(x) = x^{1/\rho} \ell(x), \) where \( \ell \in \mathcal{S} \mathcal{V}. \) Finally, set \( b_n = b(n), n \in \mathbb{N}. \) If
\[
nP(|X| > b_n) < \infty \quad \text{as} \quad n \to \infty. \tag{2.1}
\]
Then
\[
\limsup_n \frac{1}{a_n} \sum_{k=1}^{n} \frac{X_k - EX_k I_{|X_k| \leq b_n}}{b_n} \leq \gamma(\omega) \quad a.s. \tag{2.2}
\]
\[
\liminf_n \frac{1}{a_n} \sum_{k=1}^{n} \frac{X_k - EX_k I_{|X_k| \leq b_n}}{b_n} \geq -\gamma(\omega) \quad a.s. \tag{2.3}
\]
where \( I_{[\cdot]} \) denotes the indicator function.

**Proof.** Let \( Y_{k,n} = X_k I_{|X_k| \leq b_n}, \mu_{k,n} = EY_{k,n}, 1 \leq k \leq n, n \in \mathbb{N}. \) Thus we have
\[
\sum_{k=1}^{n} P(X_k \neq Y_{k,n}) = \sum_{k=1}^{n} P(|X_k| > b_n)
\]
\[
\leq C \sum_{k=1}^{n} P(|X| > b_n) < \infty
\]
which implies
\[
\sum_{k=1}^{n} \frac{X_k - Y_{k,n}}{b_n} \text{ converges a.s.}
\]
and hence
\[
\frac{1}{a_n} \sum_{k=1}^{n} \frac{X_k - Y_{k,n}}{b_n} \to 0 \quad a.s. \tag{2.4}
\]
Putting \( \lambda = \pm 1, \) set
\[
Q_{k,n}(\lambda) = E[\exp(\lambda \frac{Y_{k,n} - \mu_{k,n}}{b_n})] = \int_{|x_k| \leq |b_n|} p_k(x_k) \exp[\frac{\lambda(x_k - \mu_{k,n})}{b_n}] dx_k.
\]
\[
p_n(\lambda; x_k) = \frac{p_k(x_k) \exp[\lambda((x_k - \mu_{k,n})/b_n)] I_{[|x_k| \leq |b_n|]} Q_{k,n}(\lambda)}{Q_{k,n}(\lambda)}.
\]
We define the likelihood ratio as below
\[ \Lambda_n(\lambda; \omega) = \frac{q_n(\lambda; X_1, \cdots, X_n)}{p_n(X_1, \cdots, X_n)}. \]

The lemma can be rewrite as
\[ \limsup_n a_n^{-1} \log \Lambda_n(\lambda; \omega) \leq 0, \ a.s. \]

Thus we have
\[ \limsup_n \frac{1}{a_n} [\lambda \sum_{k=1}^{n} (Y_{k,n} - \mu_{k,n}) + \gamma_n(\omega) - \sum_{k=1}^{n} \log Q_{k,n}(\lambda)] \leq 0, \ a.s. \quad (2.5) \]

(1.4) and (2.5) imply
\[ \limsup_n \frac{\lambda}{a_n} \sum_{k=1}^{n} (Y_{k,n} - \mu_{k,n}) \leq \gamma(\omega) + \limsup_n \frac{1}{a_n} \sum_{k=1}^{n} \log Q_{k,n}(\lambda), \ a.s. \]

From the inequality \( 0 \leq e^x - 1 - x \leq \frac{1}{2} x^2 e^{|x|} \quad \text{for all } x \in R \) and note that \( \{ \frac{Y_{k,n} - \mu_{k,n}}{b_n} \} \) are uniformly bounded(by 2) random variables, we have
\[ 0 \leq Q_k(\lambda) - 1 = E[\exp\{\lambda(Y_{k,n} - \mu_{k,n}) - 1 - \lambda(\frac{Y_{k,n} - \mu_{k,n}}{b_n})\}] \]
\[ \leq \frac{1}{2} \lambda^2 e^{2|\lambda|} E\left[\frac{Y_{k,n} - \mu_{k,n}}{b_n}\right]^2 \leq \frac{1}{2} \lambda^2 e^{2|\lambda|} E\left[\frac{Y_{k,n}}{b_n}\right]^2. \quad (2.6) \]

Since
\[
\frac{1}{b_n^2} \sum_{k=1}^{n} E[X_k^2 I_{\{|X_k| \leq b_n\}}] \\
= \frac{1}{b_n^2} \sum_{k=1}^{n} \sum_{j=1}^{n} E[X_k^2 I_{\{|b_{j-1} < |X_k| \leq b_j\}}] \\
\leq \frac{1}{b_n^2} \sum_{k=1}^{n} \sum_{j=1}^{n} b_j^2 I_{\{|b_{j-1} < |X_k| \leq b_j\}} \\
= \frac{1}{b_n^2} \sum_{k=1}^{n} \sum_{j=1}^{n} j^{2/p}(\ell(j))^2 P(b_{j-1} < |X_k| \leq b_j) \\
\leq C \frac{1}{b_n^2} \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{j} j^{2/p-1}(\ell(i))^2 P(b_{j-1} < |X_k| \leq b_j) \\
= C \frac{1}{b_n^2} \sum_{k=1}^{n} \sum_{j=1}^{n} j^{2/p-1}(\ell(j))^2 \sum_{i=j}^{n} P(b_{i-1} < |X_k| \leq b_i)
\]
\[
C \frac{1}{b^2_n} \sum_{k=1}^{n} \sum_{j=1}^{n} j^{2/p-1}(\ell(j))^2 P(b_{j-1} < |X_k| \leq b_n)
\]
\[
\leq C \frac{1}{b^2_n} \sum_{k=1}^{n} \sum_{j=1}^{n} j^{2/p-1}(\ell(j))^2 P(|X_k| > b_{j-1})
\]
\[
\leq C \frac{1}{b^2_n} \sum_{k=1}^{n} \sum_{j=1}^{n} j^{2/p-1}(\ell(j))^2 P(|X| > b_{j-1})
\]
\[
\leq C \frac{1}{n^{2/p-1}(\ell(n))^2} \sum_{j=0}^{n-1} j^{2/p-2}(\ell(j))^2 j P(|X| > b_j)
\]
which converges as \(n \to \infty\). The convergence is justified, since we are faced with a weighted average of quantities that converge; the weights are \(j^{(2/p) - 2}(\ell(j))^2\), the sum of which behave (although \(O\) is enough) like
\[
\sum_{j=1}^{n} j^{(2/p) - 2}(\ell(j))^2 \sim \frac{\rho}{2 - \rho} n^{(2/p) - 1}(\ell(n))^2 \quad \text{a.s.} \quad n \to \infty.
\]
Clearly
\[
0 \leq \limsup_n \frac{1}{a_n} \sum_{k=1}^{n} [Q_k(\lambda) - 1]
\]
\[
\leq \frac{1}{2} \lambda^2 e^{2|\lambda|} \limsup_n \frac{1}{a_n} \sum_{j=1}^{n} C \frac{1}{n^{2/p-1}(\ell(n))^2} \sum_{j=0}^{n-1} j^{2/p-2}(\ell(j))^2 j P(|X| > b_j)
\]
\[
= 0 \quad \text{a.s.} \quad \text{(2.7)}
\]
(2.7) together with the inequality \(0 \leq \log x \leq x - 1 \quad (x \geq 1)\) yield
\[
0 \leq \limsup_n a_n^{-1} \sum_{k=1}^{n} \log Q_{k,n}(\lambda) \leq 0, \quad \text{a.s.}
\]
Hence we have
\[
\limsup_n \frac{\lambda}{a_n} \sum_{k=1}^{n} \left[\frac{Y_{k,n} - \mu_{k,n}}{b_n}\right] \leq \gamma(\omega) \quad \text{a.s.} \quad \text{(2.8)}
\]
Let \(\lambda = 1\), we have by (2.8)
\[
\limsup_n \frac{1}{a_n} \sum_{k=1}^{n} \left[\frac{Y_{k,n} - \mu_{k,n}}{b_n}\right] \leq \gamma(\omega) \quad \text{a.s.} \quad \text{(2.9)}
\]
Let \(\lambda = -1\), we have
\[
\liminf_n \frac{1}{a_n} \sum_{k=1}^{n} \left[\frac{Y_{k,n} - \mu_{k,n}}{b_n}\right] \geq -\gamma(\omega) \quad \text{a.s.} \quad \text{(2.10)}
\]
Noting that
\[
\frac{X_k - \mu_{k,n}}{b_n} = \frac{(X_k - Y_{k,n}) + (Y_{k,n} - \mu_{k,n})}{b_n}
\]
(2.2) and (2.3) follows immediately from (2.8),(2.9) and (2.4)

\[\square\]

**Theorem 2.** Under the conditions of Theorem 1, if \(\{X_n, n \in \mathbb{N}\} \prec X(C)\), then (2.2) and (2.3) hold.

**References**


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