Approximation of Random Fixed Points of Non-self Asymptotically Nonexpansive Random Mappings¹

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Abstract

In this paper, we introduce a random iteration process, and prove that the iteration process converges to a random fixed point of non-self asymptotically nonexpansive random mappings in real uniformly convex separable Banach spaces. The results presented in this paper are new for random mappings.

Keywords: Non-self asymptotically nonexpansive random mapping; Random fixed point; Iteration process; Separable Banach space

1 Introduction

Let K be a nonempty closed convex subset of real normed linear space E . A self-mapping $T : K \rightarrow K$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in K$. A self-mapping $T : K \rightarrow K$ is called asymptotically nonexpansive if there exists sequences $\{k_n\} \subset [1, \infty)$, $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \quad (1.1)$$

for all $x, y \in K$ and each $n \geq 1$. A mapping $T : K \rightarrow K$ is said to be uniformly L-Lipschitzian if there exists constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\| \quad (1.2)$$

for $\forall x, y \in K$ and each $n \geq 1$.

Being an important generalization of the class of asymptotically nonexpansive self-mappings, the concept of deterministic non-self asymptotically nonexpansive mappings was introduced by Chidume, Ofoedu and Zegeye[4] in 2003. The non-self asymptotically nonexpansive mapping is defined as follows:

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Definition 1.1[4]. Let K be a nonempty subset of a real normed linear space E . Let $P : E \rightarrow K$ be the nonexpansive retraction of E onto K . A non-self mapping $T : K \rightarrow E$ is called asymptotically nonexpansive if there exists sequence $\{k_n\} \subset [1, \infty)$, $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n \|x - y\| \quad (1.3)$$

for all $x, y \in K$ and each $n \geq 1$. $T : K \rightarrow E$ is said to be uniformly L-Lipschitzian if there exists constant $L > 0$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L \|x - y\| \quad (1.4)$$

for all $x, y \in K$ and each $n \geq 1$.

By studying the following iteration process:

$$x_1 \in K, \quad x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad (1.5)$$

Chidume, Ofoedu and Zegeye [4] got some strong convergence theorems for non-self asymptotically nonexpansive mapping.

In 2006, Wang [14] further generalized the iteration scheme as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}y_n), \\ y_n = P((1 - \beta_n)x_n + \beta_n T(PT)^{n-1}x_n), n \geq 1, \end{cases} \quad (1.6)$$

and got some new results.

Remark 1.1. If T is a self mapping, then P becomes the identity mapping. Thus (1.3) and (1.4) reduce to (1.1) and (1.2), respectively.

Remark 1.2. As $\beta_n = 0$ for all $n \geq 1$, the iteration scheme (1.6) reduce to (1.5).

The theory of random operator is an important branch of probabilistic analysis which plays a key role in many applied areas. The study of random fixed points forms a central topic in this area. Research of this direction was initiated by Prague School of Probabilists in connection with random operator theory. Random fixed point theory has attracted much attention in recent times since the publication of the survey article by Bharucha-Reid [1] in 1976, in which the stochastic versions of some well-known fixed point theorems were proved. A lot of efforts have been devoted to random fixed point theory and applications (see e.g., [2, 3, 5-11, 15, etc]) .

In recently years, many results about deterministic nonexpansive self-mappings and asymptotically nonexpansive self-mappings have been randomized by some authors([3] and therein). The purpose of this paper is to construct a random iteration scheme to approximate random fixed points of non-self asymptotically nonexpansive mapping and to show some convergence theorems for such mappings in uniformly convex separable Banach spaces.

2 preliminaries

Let (Ω, Σ) be a measurable space (Σ -sigma algebra) and K a nonempty subset of a real Banach space E . A mapping $\xi : \Omega \rightarrow K$ is said to be measurable if $\xi^{-1}(U \cap K) \in \Sigma$ for every Borel subset U of E . A mapping $T : \Omega \times K \rightarrow K$ is said to be a random mapping if for each fixed $x \in K$, the mapping $T(\cdot, x) : \Omega \rightarrow K$ is measurable. A measurable mapping $\xi^* : \Omega \rightarrow K$ is called a random fixed point of the random mapping $T : \Omega \times K \rightarrow K$ if $T(\omega, \xi^*(\omega)) = \xi^*(\omega)$, for each $\omega \in \Omega$.

Throughout of this paper, we denote the set of all random fixed points of a random mapping T by $RF(T)$.

Definition 2.1. A subset K of E is said to be retract if there exists continuous mapping $P : E \rightarrow K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : E \rightarrow E$ is said to be a retraction if $P^2 = P$.

Note. If a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, range of P .

Definition 2.2. Let K be a nonempty closed convex subset of a real uniformly convex separable Banach space E and $T : \Omega \times K \rightarrow E$ be a non-self random mapping. Then the random mapping T is said to be a

(1) non-self asymptotically nonexpansive random mapping if there exists a measurable mapping sequence $k_n : \Omega \rightarrow [1, +\infty)$ with $\lim_{n \rightarrow \infty} k_n(\omega) = 1$ for each $\omega \in \Omega$, such that for arbitrary $x, y \in K$ and each $\omega \in \Omega$,

$$\|T(PT)^{n-1}(\omega, x) - T(PT)^{n-1}(\omega, y)\| \leq k_n(\omega)\|x - y\|,$$

where, $n = 1, 2, \dots$.

(2) uniformly L-Lipschitzian random mapping if there exists constant $L > 0$ such that for arbitrary $x, y \in K$ and each $\omega \in \Omega$

$$\|T(PT)^{n-1}(\omega, x) - T(PT)^{n-1}(\omega, y)\| \leq L\|x - y\|,$$

where, $n = 1, 2, \dots$.

(3) demicompact random mapping if for a sequence of measurable mappings $\{\xi_n\}$ from Ω to K , with $\lim_{n \rightarrow \infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0$ for each $\omega \in \Omega$, there exists a subsequence $\{\xi_{n_j}\}$ of $\{\xi_n\}$ such that $\xi_{n_j}(\omega) \rightarrow \xi(\omega)$ as $j \rightarrow \infty$, for each $\omega \in \Omega$, where ξ is a measurable mapping from Ω to K .

(4) completely continuous random mapping if the sequence $\{x_n\}$ in K converges weakly to x_0 implies that $\{T(\omega, x_n)\}$ converges strongly to $T(\omega, x_0)$ for each $\omega \in \Omega$.

Remark 2.1. As a matter of fact, every non-self asymptotically nonexpansive random mapping is uniformly L-Lipschitzian, where $L = \sup_{\omega \in \Omega, n \geq 1} k_n(\omega)$.

Definition 2.3. Let $T : \Omega \times K \rightarrow E$ be a non-self random mapping, where K is a nonempty convex subset of a separable real uniformly convex Banach space E . The random iteration scheme is defined as follows:

$$\begin{aligned}\xi_{n+1}(\omega) &= P((1 - \alpha_n)\xi_n(\omega) + \alpha_n T(PT)^{n-1}(\omega, \eta_n(\omega))), \\ \eta_n(\omega) &= P((1 - \beta_n)\xi_n(\omega) + \beta_n T(PT)^{n-1}(\omega, \xi_n(\omega))), \quad n \geq 1,\end{aligned}\quad (2.1)$$

where $0 \leq \alpha_n, \beta_n < 1$ and $\xi_1 : \Omega \rightarrow K$ is an arbitrary given measurable mapping from Ω to K , P is a nonexpansive retraction from E to K .

In fact, as $\beta_n = 0$ for any $n \geq 1$, the iteration scheme (2.1) reduces to the following random iteration scheme:

$$\xi_{n+1}(\omega) = P((1 - \alpha_n)\xi_n(\omega) + \alpha_n T(PT)^{n-1}(\omega, \xi_n(\omega))).$$

In our paper, we discuss (2.1) only.

Obviously, the sequences $\{\xi_n\}$ and $\{\eta_n\}$ are two measurable sequences from Ω to K .

We restate the following lemmas which play crucial roles in our proofs.

Lemma 2.1[13]. Let $\{u_n\}$ and $\{v_n\}$ be two sequences of nonnegative real numbers such that $u_{n+1} \leq u_n + v_n$, $\forall n \geq 1$ and $\sum_{n=1}^{\infty} v_n < \infty$. Then $\lim_{n \rightarrow \infty} u_n$ exists. In addition, if there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \rightarrow 0$ as $j \rightarrow \infty$, then $\lim_{n \rightarrow \infty} u_n = 0$.

Lemma 2.2[12]. Let E be a real uniformly convex Banach space and $0 \leq p \leq t_n \leq q < 1$ for all positive integer $n \geq 1$. Also suppose $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$ and $\limsup_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 2.3[4]. Let E be a real uniformly convex Banach space, K a nonempty closed subset of E , and let $T : K \rightarrow E$ be a non-self asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \rightarrow 1$ as $n \rightarrow \infty$. Then $I - T$ is demiclosed at zero.

3 Main Results

Lemma 3.1. Let E be a real uniformly convex separable Banach space, K a nonempty closed convex subset of E . Suppose that $T : \Omega \times K \rightarrow K$ is a non-self asymptotically nonexpansive random mapping with a measurable mapping sequence $k_n : \Omega \rightarrow [1, +\infty)$ such that $\lim_{n \rightarrow \infty} k_n(\omega) = 1$ and $\sum_{n=1}^{\infty} (k_n(\omega) - 1) < \infty$ for each $\omega \in \Omega$. Assume that $RF(T) \neq \phi$ and the sequence of mappings $\{\xi_n\}$ is generated by (2.1). Then

(1) for each $\omega \in \Omega$, there exists $M(\omega) > 0$ such that $\|\xi_n(\omega) - \xi(\omega)\| \leq M(\omega)$, $n = 1, 2, \dots$,

(2) $\lim_{n \rightarrow \infty} \|\xi_n(\omega) - \xi^*(\omega)\|$ exists, for each $\omega \in \Omega$ and any $\xi^* \in RF(T)$.

Proof. (1). for each $\omega \in \Omega$, setting $k_n(\omega) = 1 + u_n(\omega)$. Since $\sum_{n=1}^{\infty} (k_n(\omega) - 1) < \infty$, so $\sum_{n=1}^{\infty} u_n(\omega) < \infty$. For any $\xi^* \in RF(T)$, It follows from (2.1) that

$$\begin{aligned} \|\xi_{n+1}(\omega) - \xi^*(\omega)\| &= \|P((1 - \alpha_n)(\xi_n(\omega) - \xi^*(\omega)) + \alpha_n T(PT)^{n-1}(\omega, \eta_n(\omega))) - P\xi^*(\omega)\| \\ &\leq (1 - \alpha_n)\|\xi_n(\omega) - \xi^*(\omega)\| + \alpha_n(1 + u_n(\omega))\|\eta_n(\omega) - \xi^*(\omega)\|, \end{aligned}$$

where

$$\begin{aligned} \|\eta_n(\omega) - \xi^*(\omega)\| &\leq \|(1 - \beta_n)(\xi_n(\omega) - \xi^*(\omega)) + \beta_n(T(PT)^{n-1}(\omega, \xi_n(\omega))) - \xi^*(\omega)\| \\ &\leq (1 - \beta_n)\|\xi_n(\omega) - \xi^*(\omega)\| + \alpha_n(1 + u_n(\omega))\|\xi_n(\omega) - \xi^*(\omega)\| \\ &< (1 + u_n)\|\xi_n(\omega) - \xi^*(\omega)\|. \end{aligned}$$

Thus

$$\begin{aligned} \|\xi_{n+1}(\omega) - \xi^*(\omega)\| &\leq (1 - \alpha_n)\|\xi_n(\omega) - \xi^*(\omega)\| + \alpha_n(1 + u_n(\omega))\|\eta_n(\omega) - \xi^*(\omega)\| \\ &< (1 - \alpha_n)\|\xi_n(\omega) - \xi^*(\omega)\| + \alpha_n(1 + u_n(\omega))(1 + u_n(\omega))\|\xi_n(\omega) - \xi^*(\omega)\| \\ &= [1 + \alpha_n(2u_n(\omega) + u_n^2(\omega))]\|\xi_n(\omega) - \xi^*(\omega)\| \\ &< (1 + 2u_n(\omega) + u_n^2(\omega))\|\xi_n(\omega) - \xi^*(\omega)\| \\ &< \sum_{n=1}^{\infty} (2u_n(\omega) + u_n^2(\omega)) \|\xi_1(\omega) - \xi^*(\omega)\|. \end{aligned}$$

Since $\sum_{n=1}^{\infty} u_n(\omega) < \infty$ for each $\omega \in \Omega$, so for each $\omega \in \Omega$, we can set $M(\omega) = \sum_{n=1}^{\infty} (2u_n(\omega) + u_n^2(\omega)) \|\xi_1(\omega) - \xi^*(\omega)\|$. Thus, we obtain that for each $\omega \in \Omega$, $\|\xi_n(\omega) - \xi^*(\omega)\| \leq M(\omega)$. This implies that $\{\xi_n(\omega)\}$ is bounded for each $\omega \in \Omega$.

(2). By the process and result of (1), for each $\omega \in \Omega$, we have

$$\|\xi_{n+1}(\omega) - \xi^*(\omega)\| \leq \|\xi_n(\omega) - \xi^*(\omega)\| + (2u_n + u_n^2)M.$$

It follows from Lemma 2.1 that $\lim_{n \rightarrow \infty} \|\xi_n(\omega) - \xi^*(\omega)\|$ exists, for each $\omega \in \Omega$ and any $\xi^* \in RF(T)$.

Lemma 3.2. Let E be a real uniformly convex separable Banach space, K a nonempty closed convex subset of E . Suppose that $T : \Omega \times K \rightarrow K$ is a non-self asymptotically nonexpansive random mapping with a measurable mapping sequence $k_n : \Omega \rightarrow [1, +\infty)$ such that $\lim_{n \rightarrow \infty} k_n(\omega) = 1$ and $\sum_{n=1}^{\infty} (k_n(\omega) - 1) < \infty$ for each $\omega \in \Omega$. Assume that $RF(T) \neq \emptyset$ and the sequence of mappings $\{\xi_n\}$ is generated by (2.1). Then $\lim_{n \rightarrow \infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0$.

Proof. For each $\omega \in \Omega$, setting $k_n(\omega) = 1 + u_n(\omega)$. For any $\xi^* \in RF(T)$, it follows from Lemma 3.1 that $\lim_{n \rightarrow \infty} \|\xi_n(\omega) - \xi^*(\omega)\|$ exists for each $\omega \in \Omega$. Assume $\lim_{n \rightarrow \infty} \|\xi_n(\omega) - \xi^*(\omega)\| = c$. From (2.1), we have

$$\|\eta_n(\omega) - \xi^*(\omega)\| \leq (1 + u_n(\omega))\|\xi_n(\omega) - \xi^*(\omega)\|. \quad (3.1)$$

Taking *limsup* on both sides in (3.1), we have

$$\limsup_{n \rightarrow \infty} \|\eta_n(\omega) - \xi^*(\omega)\| \leq c. \quad (3.2)$$

In addition, $\|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi^*(\omega)\| \leq k_n(\omega)\|\eta_n(\omega) - \xi^*(\omega)\|$, taking *limsup* on both sides in this inequality, we have

$$\limsup_{n \rightarrow \infty} \|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi^*(\omega)\| \leq c. \quad (3.3)$$

Since $\lim_{n \rightarrow \infty} \|\xi_n(\omega) - \xi^*(\omega)\| = c$, then by

$$\|\xi_{n+1}(\omega) - \xi^*(\omega)\| \leq \|(1 - \alpha_n)(\xi_n(\omega) - \xi^*(\omega)) + \alpha_n(T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi^*(\omega))\|,$$

we have

$$\liminf_{n \rightarrow \infty} \|(1 - \alpha_n)(\xi_n(\omega) - \xi^*(\omega)) + \alpha_n(T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi^*(\omega))\| \geq c. \quad (3.4)$$

In addition,

$$\begin{aligned} & \|(1 - \alpha_n)(\xi_n(\omega) - \xi^*(\omega)) + \alpha_n(T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi^*(\omega))\| \\ & \leq (1 - \alpha_n)\|\xi_n(\omega) - \xi^*(\omega)\| + \alpha_n(1 + u_n(\omega))\|\eta_n(\omega) - \xi^*(\omega)\|. \end{aligned}$$

Taking *limsup* on both sides in equality above, we obtain

$$\limsup_{n \rightarrow \infty} \|(1 - \alpha_n)(\xi_n(\omega) - \xi^*(\omega)) + \alpha_n(T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi^*(\omega))\| \leq c. \quad (3.5)$$

It follows from (3.4) and (3.5) that

$$\limsup_{n \rightarrow \infty} \|(1 - \alpha_n)(\xi_n(\omega) - \xi^*(\omega)) + \alpha_n(T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi^*(\omega))\| = c. \quad (3.6)$$

By Lemma 2.2, we have

$$\lim_{n \rightarrow \infty} \|\xi_n(\omega) - T(PT)^{n-1}(\omega, \eta_n(\omega))\| = 0. \quad (3.7)$$

In addition,

$$\begin{aligned} \|\xi_n(\omega) - \xi^*(\omega)\| & \leq \|\xi_n(\omega) - T(PT)^{n-1}(\omega, \eta_n(\omega))\| + \|T(PT)^{n-1}(\omega, \eta_n(\omega)) - \xi^*(\omega)\| \\ & \leq \|\xi_n(\omega) - T(PT)^{n-1}(\omega, \eta_n(\omega))\| + (1 + u_n(\omega))\|\eta_n(\omega) - \xi^*(\omega)\|. \end{aligned}$$

Taking *liminf* on both sides in equality above, by (3.7), we have

$$\liminf_{n \rightarrow \infty} \|\eta_n(\omega) - \xi^*(\omega)\| \geq c. \quad (3.8)$$

Thus, it follows from (3.2) and (3.8) that $\lim_{n \rightarrow \infty} \|\eta_n(\omega) - \xi^*(\omega)\| = c$. It implies that $\lim_{n \rightarrow \infty} \|(1 - \beta_n)(\xi_n(\omega) - \xi^*(\omega)) + \beta_n(T(PT)^{n-1}(\omega, \xi_n(\omega))) - \xi^*(\omega)\| = c$. Then by Lemma 2.2, we obtain

$$\lim_{n \rightarrow \infty} \|\xi_n(\omega) - T(PT)^{n-1}(\omega, \xi_n(\omega))\| = 0. \quad (3.9)$$

Since $\|\xi_n(\omega) - \eta_n(\omega)\| = \|\xi_n(\omega) - P((1 - \beta_n)(\xi_n(\omega) + \beta_n T(PT)^{n-1}(\omega, \xi_n(\omega))))\| \leq \beta_n \|\xi_n(\omega) - T(PT)^{n-1}(\omega, \xi_n(\omega))\|$, so,

$$\lim_{n \rightarrow \infty} \|\xi_n(\omega) - \eta_n(\omega)\| = 0. \quad (3.10)$$

Thus

$$\begin{aligned} & \lim_{n \rightarrow \infty} \|\eta_n(\omega) - T(PT)^{n-1}(\omega, \xi_n(\omega))\| \\ &= \lim_{n \rightarrow \infty} \|\eta_n(\omega) - \xi_n(\omega) + \xi_n(\omega) - T(PT)^{n-1}(\omega, \xi_n(\omega))\| = 0. \end{aligned} \quad (3.11)$$

Since $\|\xi_{n+1}(\omega) - \xi_n(\omega)\| \leq \alpha_n \|\xi_n(\omega) - T(PT)^{n-1}(\omega, \eta_n(\omega))\|$, by (3.7), we have $\lim_{n \rightarrow \infty} \|\xi_{n+1}(\omega) - \xi_n(\omega)\| = 0$. Thus, from (3.7) and $\|\xi_{n+1}(\omega) - T(PT)^{n-1}(\omega, \eta_n(\omega))\| = \|\xi_{n+1}(\omega) - \xi_n(\omega) + \xi_n(\omega) - T(PT)^{n-1}(\omega, \eta_n(\omega))\| \leq \|\xi_{n+1}(\omega) - \xi_n(\omega)\| + \|\xi_n(\omega) - T(PT)^{n-1}(\omega, \eta_n(\omega))\|$, we have

$$\lim_{n \rightarrow \infty} \|\xi_{n+1}(\omega) - T(PT)^{n-1}(\omega, \eta_n(\omega))\| = 0. \quad (3.12)$$

It follows from (3.12) that

$$\lim_{n \rightarrow \infty} \|T(PT)^{n-2}(\omega, \eta_{n-1}(\omega)) - \xi_n(\omega)\| = 0. \quad (3.13)$$

In addition,

$$\begin{aligned} \|\xi_{n+1}(\omega) - \eta_n(\omega)\| &= \|\xi_{n+1}(\omega) - T(PT)^{n-1}(\omega, \eta_n(\omega)) + T(PT)^{n-1}(\omega, \eta_n(\omega)) - \eta_n(\omega)\| \\ &\leq \|\xi_{n+1}(\omega) - T(PT)^{n-1}(\omega, \eta_n(\omega))\| + \|\eta_n(\omega) - \xi_n(\omega)\| \\ &\quad + \|\xi_n(\omega) - T(PT)^{n-1}(\omega, \eta_n(\omega))\|. \end{aligned}$$

It follow from (3.9), (3.10) and (3.12) that

$$\lim_{n \rightarrow \infty} \|\xi_{n+1}(\omega) - \eta_n(\omega)\| = 0. \quad (3.14)$$

Since T is an asymptotically nonexpansive random mapping, T is uniformly L-Lipschitzian for some $L > 0$. Hence

$$\begin{aligned} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| &= \|\xi_n(\omega) - T(PT)^{n-1}(\omega, \xi_n(\omega)) + T(PT)^{n-1}(\omega, \xi_n(\omega)) - T\xi_n(\omega)\| \\ &\leq \|\xi_n(\omega) - T(PT)^{n-1}(\omega, \xi_n(\omega))\| + \|T(PT)^{n-1}(\omega, \eta_{n-1}(\omega)) - T(\omega, \xi_n(\omega))\| \\ &\quad + \|T(PT)^{n-1}(\omega, \xi_n(\omega)) - T(PT)^{n-1}(\omega, \eta_{n-1}(\omega))\| \\ &\leq \|\xi_n(\omega) - T(PT)^{n-1}(\omega, \xi_n(\omega))\| + k_n(\omega) \|\xi_n(\omega) - \eta_{n-1}(\omega)\| \\ &\quad + L \|T(PT)^{n-2}(\omega, \eta_{n-1}(\omega)) - \xi_n(\omega)\|. \end{aligned}$$

It follows from (3.9), (3.13) and (3.14) that $\lim_{n \rightarrow \infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0$. The proof is completed.

Theorem 3.3. Let E be a real uniformly convex separable Banach space, K a nonempty closed convex subset of E . Suppose that $T : \Omega \times K \rightarrow E$ is a non-self asymptotically nonexpansive random mapping with a measurable mapping sequence $k_n : \Omega \rightarrow [1, +\infty)$ such that $\lim_{n \rightarrow \infty} k_n(\omega) = 1$ and $\sum_{n=1}^{\infty} (k_n(\omega) - 1) < \infty$ for each $\omega \in \Omega$. Let $\{\xi_n(\omega)\}$ be generated by (2.1), where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[\varepsilon, 1 - \varepsilon]$ for some $\varepsilon > 0$. If $RF(T) \neq \phi$ and T is completely continuous, then the sequence $\{\xi_n\}$ converges to a random fixed point of T .

Proof. By Lemma 3.1, $\{\xi_n\}$ is bounded. In addition, by Lemma 3.2, $\lim_{n \rightarrow \infty} \|\xi_n - T(\omega, \xi_n(\omega))\| = 0$. Then $\{T(\omega, \xi_n(\omega))\}$ also is bounded for each $\omega \in \Omega$. Since T is completely continuous, there exists subsequence $\{T(\omega, \xi_{n_j}(\omega))\}$ of $\{T(\omega, \xi_n(\omega))\}$ such that $T(\omega, \xi_{n_j}(\omega)) \rightarrow \xi^*(\omega)$ as $j \rightarrow \infty$. It follows from Lemma 3.2 that $\lim_{j \rightarrow \infty} \|\xi_{n_j}(\omega) - T(\omega, \xi_{n_j}(\omega))\| = 0$. By Lemma 2.3, we have $\xi^*(\omega) = T(\omega, \xi^*(\omega))$ for each $\omega \in \Omega$. Since $\|\xi_{n_j} - \xi^*(\omega)\| \leq \|\xi_{n_j} - T(\omega, \xi_{n_j}(\omega))\| + \|T(\omega, \xi_{n_j}(\omega)) - \xi^*(\omega)\|$, we have $\lim_{j \rightarrow \infty} \|\xi_{n_j} - \xi^*(\omega)\| = 0$. In addition, by Lemma 3.1, we know that $\lim_{n \rightarrow \infty} \|\xi_n - \xi^*(\omega)\|$ exists. Thus $\lim_{n \rightarrow \infty} \|\xi_n(\omega) - \xi^*(\omega)\| = 0$ for each $\omega \in \Omega$. Since ξ^* is pointwise limit of the measurable mapping sequence $\{\xi_n\}$, $\xi^*(\omega)$ is measurable. Therefore, $\xi^*(\omega) \in RF(T)$. The proof is completed.

Theorem 3.4. Let E be a real uniformly convex separable Banach space, K a nonempty closed convex subset of E . Suppose that $T : \Omega \times K \rightarrow E$ is a non-self asymptotically nonexpansive random mapping with a measurable mapping sequence $k_n : \Omega \rightarrow [1, +\infty)$ such that $\lim_{n \rightarrow \infty} k_n(\omega) = 1$ and $\sum_{n=1}^{\infty} (k_n(\omega) - 1) < \infty$ for each $\omega \in \Omega$. Let $\{\xi_n(\omega)\}$ be generated by (2.1), where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[\varepsilon, 1 - \varepsilon]$ for some $\varepsilon > 0$. If $RF(T) \neq \phi$ and T is demicompact, then the sequence $\{\xi_n\}$ converges to a random fixed point of T .

Proof. Since T is demicompact, by Lemma 3.1 and Lemma 3.2, $\{\xi_n(\omega)\}$ is bounded and $\lim_{n \rightarrow \infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0$, then there exists subsequence $\{\xi_{n_j}\}$ of $\{\xi_n\}$ such that $\{\xi_{n_j}(\omega)\}$ converges strongly to $\xi^*(\omega)$ for each $\omega \in \Omega$. It follows from Lemma 2.3 that $\xi^*(\omega) = T(\omega, \xi^*(\omega))$. Since $\{\xi_{n_j}\}$ is a measurable mapping sequence, ξ^* is a measurable mapping, too. Therefore, $\xi^*(\omega) \in RF(T)$. Thus $\lim_{n \rightarrow \infty} \|\xi_n(\omega) - \xi^*(\omega)\|$ exists by Lemma 3.1. Since the subsequence $\{\xi_{n_j}(\omega)\}$ of $\{\xi_n(\omega)\}$ such that $\{\xi_{n_j}(\omega)\}$ converges to $\xi^*(\omega)$, then $\{\xi_n(\omega)\}$ converges to a random fixed point $\xi^*(\omega) \in RF(T)$. The proof is completed.

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