Derivations and Reverse Derivations in Semiprime Rings

Mohammad Samman

Department of Mathematical Sciences
King Fahd University of Petroleum & Minerals
Dhahran 31261, Saudi Arabia
msamman@kfupm.edu.sa

Nouf Alyamani

Department of Mathematics
Girls College of Science
Dammam, Saudi Arabia

Abstract

The notion of reverse derivation is studied and some properties are obtained. It is shown that in the class of semiprime rings, this notion coincides with the usual derivation when it maps a semiprime ring into its center. However, we provide some examples to show that it is not the case in general. Also it is shown that non-commutative prime rings do not admit a non-trivial skew commuting derivation.

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1 Preliminaries

Throughout, R denotes a ring with center $Z(R)$. We write $[x, y]$ for $xy - yx$. Recall that a ring R is called *prime* if $aRb = 0$ implies $a = 0$ or $b = 0$; and

it is called *semiprime* if $aRa = 0$ implies $a = 0$. A prime ring is obviously semiprime. An additive mapping d from R into itself is called a *derivation* if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A mapping f from R into itself is *commuting* if $[f(x), x] = 0$, and *skew commuting* if $f(x)x + xf(x) = 0$, for all $x \in R$. A considerable amount of work has been done on derivations and related maps during the last decades (see, e.g., [4-6] and references therein). Brešar and Vukman [2] have introduced the notion of a *reverse derivation* as an additive mapping d from a ring R into itself satisfying $d(xy) = d(y)x + yd(x)$, for all $x, y \in R$. Obviously, if R is commutative, then both derivation and reverse derivation are the same. In the present note, we explore more about reverse derivations. We will show that while the notions of derivation and reverse derivation do not coincide, the set of all derivations and the set of all reverse derivations on a ring R are not disjoint. Recall that a ring R is called *anticommutative* if $ab + ba = 0$ for all $a, b \in R$. We will provide some properties for reverse derivations on anticommutative rings. On the way of studying derivations and reverse derivations, we will show that if d is a skew commuting derivation on a non-commutative prime ring, then d must be trivial. One of our main aims is to show that for a semiprime ring R , any reverse derivation is in fact a derivation mapping R into its center. This, in turn, will force a prime ring with a non-trivial reverse derivation to be commutative. For motivation and a close view on reverse derivations, we provide the following examples.

2 Examples and Properties

The following three examples explore all the possibilities of the relationship between reverse derivations and derivations.

Example 2.1 Consider the ring $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in S \right\}$, where S is a ring such that $S^2 \neq 0$. Define $d : R \rightarrow R$ by

$$d\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}.$$

Then it is easy to check that d is both a derivation and a reverse derivation.

Example 2.2 Consider the ring R as in Example 2.1. Define $d : R \longrightarrow R$ by

$$d\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}.$$

It is easy to see that d is a derivation. Now, let $x, y \in R$ such that x and y are both non-zero (this is possible because $S^2 \neq 0$ by hypothesis). Then simple calculations show that $d(xy) \neq d(y)x + yd(x)$. So d is not a reverse derivation. The next example in this section will show that not every reverse derivation is a derivation.

Example 2.3 Consider the ring $R = \left\{ \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix} ; a, b, c \in \mathbb{R} \right\}$,

where $\mathbb{R}$ denotes the set of all real numbers. Define $d : R \longrightarrow R$ by

$$d\left(\begin{bmatrix} 0 & a & b & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & -c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let x, y be any elements of R , where

$$x = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 0 & e & f & g \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & -e \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Applying d , we can easily obtain

$$d(xy) = \begin{bmatrix} 0 & 0 & 0 & be - af \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = d(y)x + yd(x).$$

On the other hand, if we take the entries of the above matrices x and y as: $a = c = 1, b = 0$ and $f = g = 1, e = 0$, then

$$d(xy) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = d(x)y + xd(y).$$

Hence d is a reverse derivation but not a derivation.

Next we state some basic properties of reverse derivations which can be verified easily.

Proposition 2.4 Let d be a reverse derivation on a ring R . Then

- (i) If R is of characteristic 2, then d^2 is a usual derivation.
- (ii) If e is an idempotent, then $ed(e)e = 0$
- (iii) If e is commuting idempotent, then $d(e) = 0$. Moreover, if d is non-trivial and R is semiprime then e is the identity in R .
- (iv) If $1 \in R$, then $d(1) = 0$.
- (v) If R is anticommutative, then

$$d(xy^n) = \begin{cases} y^n d(x), & \text{if } n \text{ is even,} \\ y^{n-1} d(xy), & \text{if } n \text{ is odd.} \end{cases}$$

In particular,

$$d(x^n) = \begin{cases} x^{n-1} d(x), & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Example 2.5 Recall the ring R considered in Example 2.3. It is an anticommutative ring and hence property (v) in the above proposition can be viewed easily.

We state one more property of reverse derivations which deals with product of reverse derivations. Indeed, it is the reverse derivation version of Leibniz rule for higher derivations.

Proposition 2.6 Let d be a reverse derivation on a ring R . Then

$$d^n(xy) = \begin{cases} \sum_{r=0}^n \binom{n}{r} d^{n-r}(y) d^r(x), & \text{if } n \text{ is odd,} \\ \sum_{r=0}^n \binom{n}{r} d^{n-r}(x) d^r(y), & \text{if } n \text{ is even.} \end{cases}$$

3 Reverse derivations on Semiprime Rings

The following result shows that a reverse derivation is in fact a usual derivation on semiprime rings.

Proposition 3.1 A mapping d on a semiprime ring R is a reverse derivation if and only if it is a central derivation.

Proof. Let R be a semiprime ring and $d : R \longrightarrow R$ a mapping on R . It is clear that if d is a central derivation then d is a reverse derivation. So let us suppose that d is a reverse derivation. Then

$$d(xy^2) = d(y^2)x + y^2d(x) = (d(y)y + yd(y))x + y^2d(x);$$

that is,

$$d(xy^2) = d(y)yx + yd(y)x + y^2d(x), \text{ for all } x, y \in R. \quad (1)$$

Also,

$$d((xy)y) = d(y)xy + yd(xy) = d(y)xy + y(d(y)x + yd(x))$$

that is,

$$d(xy^2) = d(y)xy + yd(y)x + y^2d(x). \quad (2)$$

From (1) and (2), we get

$$d(y)yx = d(y)xy;$$

$$d(y)[y, x] = 0, \text{ for all } x, y \in R. \quad (3)$$

Replacing x by zx in (3) (and using (3) again), we get
 $d(y)[y, zx] = d(y)z[y, x] + d(y)[y, z]x = d(y)z[y, x] = 0$. Thus,

$$d(y)z[y, x] = 0, \text{ for all } x, y, z \in R. \quad (4)$$

On the other hand, a linearization of (3) leads to

$$d(u)[y, x] + d(y)[u, x] = 0, \text{ for all } x, y, u \in R,$$

$$d(y)[u, x] = -d(u)[y, x] = d(u)[x, y]. \quad (5)$$

Replacing z by $[u, x]zd(u)$ in (4) and using (5), we get

$$0 = d(y)[u, x]zd(u)[y, x] = -d(u)[y, x]zd(u)[y, x].$$

That is,

$$d(u)[y, x]zd(u)[y, x] = 0. \quad (6)$$

Since R is semiprime, by (6), we get

$$d(u)[y, x] = 0, \text{ for all } x, y, u \in R.$$

By [3, Lemma 1.1.8], $d(u) \in Z(R)$, for all $u \in R$. Hence $d(xy) = d(y)x + yd(x) = xd(y) + d(x)y$. This shows that d is a derivation on R which maps R into its center.

As a consequence, we get the following:

Corollary 3.2 Let R be a prime ring. If R admits a non-zero reverse derivation, then R is commutative.

Remark 3.3 In view of the above proposition one can easily notice that the ring R in Example 2.3 is not semiprime.

Proposition 3.4 Let R be a semiprime ring and let $a, b \in R$. Suppose that $d : R \longrightarrow R$ is a reverse derivation defined by $d(x) = ax + xb$. Then d is trivial.

Proof. For all $x, y \in R$, we have, $d(xy) = (ay + yb)x + y(ax + xb) = axy + xyb$. So, $a(xy - yx) + (xy - yx)b = ybx + yax$. This shows that

$$d([x, y]) = y(a + b)x, \text{ for all } x, y \in R. \quad (7)$$

Taking $x = y$ in equation (7), we get

$$x(a + b)x = 0. \quad (8)$$

Hence, $(x+y)(a+b)(x+y) = 0$ for all $x, y \in R$. That is, $x(a+b)y + y(a+b)x = 0$, for all $x, y \in R$. Replacing x by xy , we get $xy(a+b)y + y(a+b)xy = 0$, which from (8) will yield $y(a+b)xy = 0$. So, $y(a+b)xy(a+b) = 0$, for all $x, y \in R$. Since R is semiprime, we have $y(a+b) = 0$, for all $y \in R$. Thus, $(a+b)y(a+b) = 0$, for all $y \in R$. Again the semiprimeness of R implies that $(a+b) = 0$. Now equation (7) becomes $d([x, y]) = 0$, for all $x, y \in R$. Hence, $a[x, y] = [x, y]a$, for all $x, y \in R$. By [3, Lemma 1.1.8], we have $a \in Z(R)$, that is, $d = 0$.

Lemma 3.5 Let R be a semiprime ring and suppose that for some $a, b \in R$, the relation $ax + xb = 0$ holds for all $x \in R$. Then $a, b \in Z(R)$. Furthermore, $b = -a$.

Proof Suppose that $ax + xb = 0$. Then replacing xy for x , we get

$$axy + xyb = 0. \quad (9)$$

Multiplying the equation given in the hypothesis by y from right, we get

$$axy + xby = 0. \quad (10)$$

From equations (9) and (10) we get $xyb - xby = 0$. That is, for b and for all $x, y \in R$, we have

$$x(yb - by) = 0. \quad (11)$$

From (11) we can deduce that $b \in Z(R)$. Thus, by hypothesis, we have $ax + bx = 0$, for all $x \in R$. That is $(a + b)x = 0$, for all $x \in R$. Hence $a = -b$ is in the center of R .

As a consequence, we get a special case of [5, Theorem 2.2] as a corollary of Lemma 3.5.

Corollary 3.6 Let R be a semiprime ring and let f and g be derivations on R satisfying $f(x)y + yg(x) = 0$, for all $x, y \in R$. Then f and g map R into its center.

In view of [1, Theorem 4] and Lemma 3.5, we have the following:

Corollary 3.7 Let R be a non-commutative prime ring and let d be a skew commuting derivation on R . Then $d = 0$.

Proof By Lemma 3.5, $d(x) \in Z(R)$, for all $x \in R$. That is d is trivially centralizing on R . Hence, by [1, Theorem 4], $d = 0$.

Corollary 3.8 Let R be an anticommutative semiprime ring then R is commutative and of characteristic 2.

Proof. By hypothesis $yx + xy = 0$, for all $x, y \in R$, in particular for all $x \in R$. By Lemma 3.5, $y \in Z(R)$, for all $y \in R$. Hence R is commutative. That R is of characteristic 2 is clear.

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