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A New Computational Approach for Nonlinear Equations

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Abstract

The aim of this paper is to construct a new iterative method to solve nonlinear equations. The new method is based on Newton-Raphson method and power series. The convergence of this new scheme is addressed and a cubic order of convergence, at least, is established. To illustrate the method some examples, mainly from references have been presented, so one would be able to compare the results of the same problems obtained by applying different methods, and the advantage of the new method can be recognized.

Keywords: Nonlinear equations; Newton-Raphson method

1 Introduction

Many different methods for solving nonlinear equations are presented so far. We compare the results of our new method with the results obtained by other well known methods that are presented in [2-5]. The main proposal of this method is based on an improvement of the Newton-Raphson method and power series. Some of the advantages of the proposed method over existing method for solving equations, which worth to be mentioned are: fast convergence and more accuracy, in spite of less computation, in comparison with other methods.

2 New iterative Method

Considering the nonlinear equation $f(x) = 0$, and writing $f(x) = 0$ in truncated form of the Taylor expansion around the an initial approximation to the

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solution of equation, sayx_0:

$$f(x_0+h)=f(x_0)+hf^{'}(x_0)+\frac{h^2}{2}f^{''}(x_0)+o(h^3). \tag{1}$$

Suppose that$f^{'}(x_0)\neq 0$, and let's look for a value h such that$f(x_0+h)=0$. From which we derive the following second order algebraic equation for determining$h$

$$-\frac{h^2}{2}\frac{f^{''}(x_0)}{f^{'}(x_0)}+h-\frac{f^{(}x_0)}{f^{'}(x_0)}=0. \tag{2}$$

This can be solved by the known methods.
The discriminate of the Eq.(2) is $\triangle = 1-2\frac{f(x_0)f^{''}(x_0)}{f^{'2}(x_0)}$, which can be easily proved that is positive.
Lemma: If r be the solution of the equation $f(x)=0$, $f^{'}(r)\neq 0$ and $f\in C^3$ Then there is an open interval I , which contains r , and for any $x_0\in I$,$h(x_0)>0$, where
$h(x)=1-2\frac{f(x)f^{''}(x)}{f^{'2}(x)}$
Proof:Since$f\in C^3$ then h is continuous at r and $h(r)=1>0$. So one concludes that there exist an interval I containing r such that for$x_0\in I$ i.e. the inequality $\triangle>0$ holds by every x_0 which is close enough to the real solution of the equation (2).
One of the real solutions of Eq.(2) is

$$h=\frac{-1+\sqrt{1-2\frac{f(x_0)f^{''}(x_0)}{f^{'2}(x_0)}}}{\frac{f^{''}(x_0)}{f^{'}(x_0)}} \tag{3}$$

As mentioned above for appropriated $x's$ we have $2\frac{f(x)f^{''}(x)}{f^{'2}(x)}<1$.

Three first terms approximation of $\sqrt{1-2\frac{f(x_0)f^{''}(x_0)}{f^{'2}(x_0)}}$, by means of the Taylor expansion are as the following

$$\sqrt{1-2\frac{f(x_0)f^{''}(x_0)}{f^{'2}(x_0)}}\approx 1-\frac{f(x_0)f^{''}(x_0)}{f^{'2}(x_0)}-\frac{f^{''2}(x_0)f^2(x_0)}{2f^{'4}(x_0)} \tag{4}$$

By substitution Eq. (4) in Eq. (3) one derives

$$h=\frac{-1+1-\frac{f(x_0)f^{''}(x_0)}{f^{'2}(x_0)}-\frac{f^{''2}(x_0)f^2(x_0)}{2f^{'4}(x_0)}-}{\frac{f^{''}(x_0)}{f^{'}(x_0)}} \tag{5}$$

Or

$$h = -\frac{f(x_0)}{f'(x_0)} - \frac{f''(x_0)f^2(x_0)}{2f'^3(x_0)} \tag{6}$$

Considering $f(x_0)+h) = 0$, , the following iterative equation can be established for computation a series, say x_n , which convergence to the solution of equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)f^2(x_n)}{2f'^3(x_n)} \tag{7}$$

3 Convergence Analysis

The iterative equation (7) can be recognized as$x_{n+1} = g(x_n)$ in the well known fixed point iteration method and for the proof of convergence of the new method we follow the same procedures which are used for the convergence of that method, of course for a new $g(x)$.

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)f^2(x)}{2f'^3(x)} \tag{8}$$

Let's state the following theorems [5]. Since $g(x)$ in this paper is different form that, we state the proof.

Theorem1. Let r is a solution of equation$f(x) = 0$, where $f \in C^3$. If$f'(r) \neq 0$ then there exist an interval$I$ containing $r$such that for any $x_0 \in I$the iteration scheme (7) converges to the only solution of$f(x) = 0$ which is located in I.

Proof. By the hypothesis of the theorem we have$g(r) = r$ and g is continuous and differentiable at$x = r$. In fulfilling some easy computation the following results is reached:

$|g'(r)| = 0 < 1$ So, one concludes that there exist an$\epsilon > 0$ such that for$x \in (r-\epsilon, r+\epsilon), |g'(r)| < 1$. According to the fixed point theorem, the iterative scheme $x_{n+1} = g(x_n)$ converges to the unique solution. To study the order of convergence of the sequences, let remind the following definition from [1]:

Definition1. Letx_n converge to$r$. If there exist an integer$p \geq 1$ and a real positive constant C such that:

$$\lim \left|\frac{x_{n+1} - r}{(x_n - r)^p}\right| = C, \tag{9}$$

(9) Then$p$ is called the order of convergence and C is the constant of convergence of x_n.
Theorem2. Let r be the solution of the equation $f(x) = 0$,$f'(r) \neq 0$, and $f \in C^{k+2}$. If$g(r) = r, g'(r) = g''(r) = ... = g^{k-1}(r) = 0$ and $g^k(r) \neq 0$, then the order of convergence of the sequence which obtained from the iteration equation$x_{n+1} = g(x_n)$ is at least$k$.
Proof. Since $g(r) = r$, From the Taylor expansion of$g(x_n)$ nearby$r$, we have:

$$x_{n+1} - r = g(x_n) - g(r) = \frac{g'(r)}{1!}(x_n - r) + ... + \frac{g^{k-1}(r)}{(k-1)!}(x_n - r)^{k-1} + \quad (10)$$

$$\frac{g^k(r)}{(k)!}(x_n - r)^k + \frac{g^{k+1}(r)}{(k+1)!}(x_n - r)^{k+1} + ...$$

According o the hypothesis, the first non zero term of (11), is the term which contains $k'th$ derivative. By dividing two sides of (11) by$(x_n - 1)^k$ we get:

$$\frac{x_{n+1} - r}{(x_n - r)^k} = \frac{g^k(r)}{k!} + \frac{g^{k+1}(r)}{(k+1)!}(x_n - r) + ... \quad (11)$$

The statement$f \in C^{k+2}$ implies that $g \in C^k$ in a neighborhood of$x = r$.
So:
$\lim |\frac{x_{n+1}-r}{(x_n-r)^k} = \frac{|g^k(r)}{k!} \neq 0$. Since $g^k(r) \neq 0$ the order of convergence of method is$k$.
Corollary. The order of convergence of the sequence, derived by (8) is at least 3.
Proof. It can be easily verified that$g'(r) = 0, g''(r) = 0$, i.e. the order of convergence of the sequence is at least 3.

4 Numerical examples

In this part we consider three examples from references [2-5] which have been solved by different approaches, so one can be able to compare the results of these methods. Besides the examples are solved by Newton-Raphson method and the new method which have been introduced in this paper. The results are presented in the tables 1-3.
Example 1. Consider the equation$x^3 + 4x^2 + 8x + 8 = 0$, , with the exact solution$x = -2$, and $x_0 = -1$.

Table 1. The results of different methods for example 1.

Method	number of iterations	obtain solution	Error
Newton Raphson	1	-2.00000000	0
Ref.[2]	5	-1.98000000	0.02
Ref.[3]	1	-2.00000000	0
Ref.[4]	2	-2.00398774	1.00E-05
Ref.[4]	3	-2.000100907	3.98 E-06
New Method	3	-2.00000291	2.91E-07

Example 2. Let's solve the equation $x-2-e^{-x} = 0$, with the exact solution$x = 2.12002823$ and$x_0 = 0$.

Table 2.The results of different methods for example 2.

Method	number of iterations	obtain solution	Error
Newton Raphson	3	2.12002823	0
Ref.[2]	6	-1.98000000	0.02
Ref.[3]	1	2.2001330	1.49E-06
Ref.[4]	4	2.12001616	1.20E-06
Ref.[5]	2	2.12002823	0
New Method	2	2.12002823	0

Example 3. Let's consider the equation$x^2 - (1-x)^5 = 0$ with the exact solution$x = 0.34595481$, and $x_0 = 0.2$

Table 3.The results of different methods for example 3.

Method	number of iterations	obtain solution	Error
Newton Raphson	3	0.34595377	1.04E-06
Ref.[2]	10	0.34062225	5.33E-04
Ref.[3]	5	0.4602136	1.04E-06
Ref.[4]	2	0.34595464	1.69E-07
Ref.[5]	2	0.34595218	2.62 E-06
New Method	2	0.34595482	9.00E-09

5 Conclusion

In this In this paper an efficient iterative method is presented which is almost more accurate than existing methods, which have been presented by different authors to solve nonlinear equations. The convergence of method has been proved and the order of convergence is established. Advantages of the method, worth to be mentioned are fast convergence, and more accuracy in spite of less computation. The computations associated with examples were performed by using Maple 10.

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