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A Numerical Extension on a Convex Nonlinear Elliptic Problem

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Abstract

In this paper we deal with the class of quasilinear elliptic Dirichlet boundary value problem of type $-\Delta u = \lambda(\frac{1}{5} + \frac{1}{4}u^{\frac{3}{4}})$, where the effect of the super linear term allow us to find some results in existence and multiplicity of numerical positive solutions for varying $\lambda > 0$. Using finite difference method and we compare the obtained solutions with each other.

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1 Introduction

In studying in selection migration model in population genetics, biology and other fields in physics and engineering , we often meet the nonlinear problem of this type

$$\begin{cases} -\Delta u(x) = \lambda(1 + u(x))^q & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^N$ $(N \geq 1)$ with boundary $\partial\Omega$.

Some type of elliptic equations were presented in [1, 2, 3, 4, 5]. In this paper another useful numerical technique will be presented and obtained numerical solutions for

$$\begin{cases} -\Delta u(x) = \lambda(\frac{1}{5} + \frac{1}{4}u^{\frac{3}{4}}(x)) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{2}$$

We will show following proposition in numerical sense that can lead us to guess the behavior of the branch of solutions.

Proposition . Let $\lambda > 0$ then there is nonnegative solution for problem (2), More over if $0 < \lambda_1 < \lambda_2$ then $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$.

2 Finite difference method

The mathematical formulations of type (1) and (2) are models of physical, chemical and a wide usage in biological phenomena, and their use has also spread into economics, financial forecasting and other fields, that involve rate of change with respect to two or more independent variables, usually representing time, length or angle, lead either to a partial differential equation or to a set of such equations.

Special cases of the two dimensional second equation

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu + g = 0,$$

where a, b, c, d, e, f, and g may be functions of the independent variables x and y and of the dependent variable u, occur more frequently than any other, because they are often the mathematical form of one of the conservation principles of physics.

It is essential to approximate the solution of those partial differential equations numerically in order to investigate the predictions of the mathematical models, as the exact solutions are usually unavailable.

Numerical technique base on finite difference method that is one of the discretization methods, can lead us to find approximate solutions for an elliptic partial differential equation of orders such as

$$-\Delta u = f(\lambda, u).$$

Assume u is a function of the independent variables x and y. Subdivide the xot plane into sets of equal rectangles of sides $\delta_x = h$, $\delta_y = k$, by equally spaced grid lines parallel to Oy defined by $x_i = ih$, s.t. i=0,1,2,..., and equally spaced grid lines parallel to Ox defined by $y_j = jk$, s.t. j=0,1,2,....

We know if denote $u(ih, ik) = u_{i,j}$, Then

$$(u_{xx})_{i,j} \simeq \frac{u((i+1)h, jk) - 2u(ih, jk) + u((i-1)h, jk)}{h^2}$$

or

$$(u_{xx})_{i,j} \simeq \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \tag{3}$$

with a leading error of $O(h^2)$. Similarly

$$(u_{yy})_{i,j} \simeq \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \tag{4}$$

with a leading error of $O(k^2)$.

We consider problem (2) and use (3),(4) to find difference equations.Then we solve problem (2) with numerically method in two dimensions of Ω.

Case1: Let $\Omega = [0,1] \times [0,1]$, thus problem (2) will be of the form

$$\begin{cases} -(u_{xx} + u_{yy}) = \lambda(\frac{1}{5} + \frac{1}{4}u^{\frac{3}{4}}(x)) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{5}$$

From Dirichlet boundary condition :

$$u_{i,0} = u_{0,j} = 0, \qquad \forall i, j$$

where $u_{i,j} = u(x_i, y_j)$.

Using the approximation of u_{xx} and u_{yy} we have the following difference equation

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \lambda(\frac{1}{5} + \frac{1}{4}u_{i,j}^{\frac{3}{4}}(x))$$

Thus, we obtain

$$u_{i-1,j} - 2(1+r)u_{i,j} + u_{i+1,j} + ru_{i,j-1} + ru_{i,j+1} = -h^2\lambda(\frac{1}{5} + \frac{1}{4}u_{i,j}^{\frac{3}{4}}), \tag{6}$$

where $i = 1, 2, 3, ..., m-1$ and $j = 1, 2, 3, ..., n-1$, such that $n = \frac{1}{k}$ and $m = \frac{1}{h}$.

Case2: Let $\Omega = [0,1] \times [0,1] \times [0,1]$,thus problem (2) will be form of

$$u_{i-1,j,k} - 2(1+r+s)u_{i,j,k} + u_{i+1,j,k} + ru_{i,j-1,k} + ru_{i,j+1,k} + su_{i,j,k-1} + su_{i,j,k+1} =$$

$$-h^2\lambda(\frac{1}{5} + \frac{1}{4}u_{i,j,k}^{\frac{3}{4}}), \tag{7}$$

where $r = (\frac{h}{k})^2$, $s = (\frac{h}{t})^2$, $\delta z = t$, $i = 1, 2, 3, ..., m-1$, $j = 1, 2, 3, ..., n-1$, and $k = 1, 2, 3, ..., p-1$. Note that n, m, and p are the number of grid points to Ox, Oy, and Oz, in which $n = \frac{1}{k}$, $m = \frac{1}{h}$ and $p = \frac{1}{t}$.

3 Numerical results

In this section, we consider problem (2) and use results in the previous section to find numerical solutions. Consequently, we can solve the system of equations produced by difference equations of (5) and (6) with successive over relaxation iterative method.

In the Tables 1 and 2, we will show the numerical results of $\|u\|_\infty$ respective to the variation of λ in the interval $[0, 200]$ for $N = 2$ and $N = 3$ (N is dimension of Ω), which it coincides to proposition. These results obtained by using Matlab version 7 software.
In generally the relation between λ and $\|u\|_\infty$ can be illustrated in figure1.
Finally, in figure2 we have drawn the mesh of function u in the case1 for $h = 0.05$, $k = 0.05$ and $\lambda = 200$.

$2D$, $h = 0.5$, $k = 0.5$			
λ	$\|u\|_\infty$	λ	$\|u\|_\infty$
0.01	0.0122	50	24.3012
0.05	0.0222	100	84.7802
0.10	0.0350	150	220.2947
1.00	0.2718	199	491.5414
2.00	0.5426	200	499.1249

Tab.1

$3D$, $h = 0.5$, $k = 0.5$,$t = 0.1$			
λ	$\|u\|_\infty$	λ	$\|u\|_\infty$
0.01	0.0028	50	29.0490
0.05	0.0153	100	97.1603
0.1	0.0310	150	241.3573
1.00	0.3230	199	517.8779
2.00	0.6567	200	525.4950

Tab.2

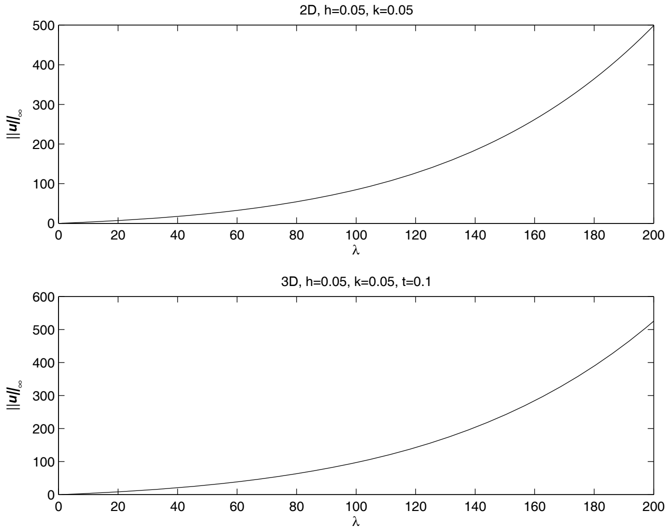

Fig. 1

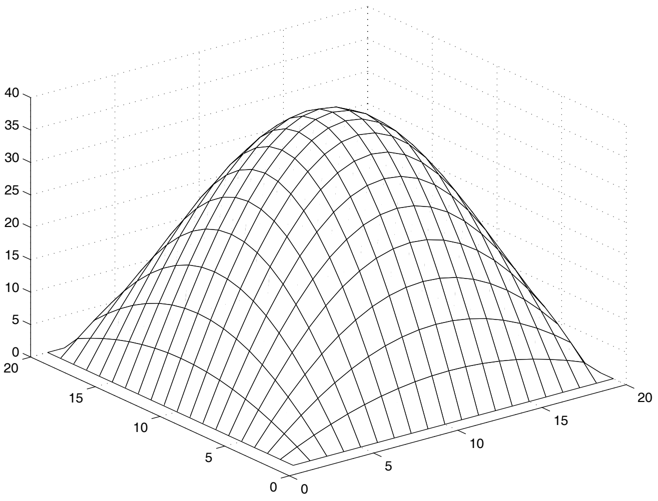

Fig. 2

4 Conclusion

In this paper we have distinguished a finite difference method for solving some of nonlinear elliptic partial differential equations which was used in section2. A numerical example in section3, confirm that approximations explained in section2 are suitable.

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