

Links Corresponding to Necklaces

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Abstract. Given any graph G which is isomorphic to an n -necklace, we give the number of components of the link corresponding to G .

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1. INTRODUCTION

There are several research problems in knot theory which continues to generate a lot of interest and activity. One problem which is still of interest is classification of knots. There are many classes of knots which have been studied in the literature, for example see [1, 2].

It has been shown in the literature that each signed planar graph has a corresponding link projection and vice-versa see [2, 3, 5]. Thus each planar graph which is not signed has a corresponding link universe. Thus we can turn problems about knots into problems about graphs. Thus one question which we can easily translate from knots to graphs is that of classifying knots. One such problem of interest is using classes of planar graphs to classify knots, see [4].

In this paper we study a class of knots corresponding to a class of planar graphs called necklaces. In particular we answer the following questions, what values of n , of an n -necklace, gives a link and what values gives a knot? If it is a link how many components are there? The component number will mean

the number of components of a link. We will use standard notation of graph theory and knot theory. For details we refer to [1, 6].

2. AN n -NECKLACE AND AN n -LADDER

In this section we begin by giving a definition and an example of an n -necklace. Then we give a definition and an example of an n -ladder. Finally we show how an n -necklace can be constructed from an $(n + 1)$ -ladder.

A graph G is called an n -necklace if it has the following set of vertices $V(G) = \{1, 2, \dots, n, n + 1, \dots, 2n\}$ and the edge set $E(G) = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}\} \cup \{\{n+1, n+2\}, \{n+2, n+3\}, \dots, \{2n-1, 2n\}, \{2n, n+1\}\} \cup \{\{1, 2n\}, \{2, 2n-1\}, \dots, \{n-1, n+2\}, \{n, n+1\}\}$. Figure 1 is an example of a 6-necklace. This graph is planar and hence we can construct a link universe

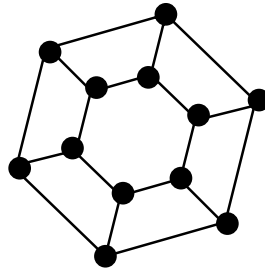


FIGURE 1. 6-necklace

corresponding to it. For the construction of a link universe from graphs we refer to [1, 2, 5]. We will talk of the link universe because the graphs are not signed. We will not bother with signs because the signs of the edges or the type of a crossing of a link does not affect the number of components of a link.

The graph we are about to define gives us an alternative way of looking at an n -necklace which will be very useful in this paper. A graph G is called an n -ladder if it has the following set of vertices $V(G) = \{1, 2, \dots, 2n\}$ and the edge set $E(G) = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}\} \cup \{\{n+1, n+2\}, \{n+2, n+3\} \dots \{2n-1, 2n\}\} \cup \{\{1, 2n\}, \{2, 2n-1\}, \dots, \{n, n+1\}\}$. The edges are of two types. An edge $e \in \{\{1, 2n\}\{2, 2n-1\}, \dots, \{n-1, n+2\}\{n, n+1\}\}$ is called a *rung*. An edge $e \in \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}\} \cup \{\{n+1, n+2\}, \{n+2, n+3\}, \dots, \{2n-1, 2n\}\}$ is called an *s-bar*. This class of graphs is planar, hence we can talk about its corresponding link universes. Figure 2 is an example of a 6-ladder.

It can be easily observed that an n -necklace can be constructed from an $(n + 1)$ -ladder by simply identifying edge $\{1, 2(n + 1)\}$ with edge $\{n + 1, n + 2\}$,

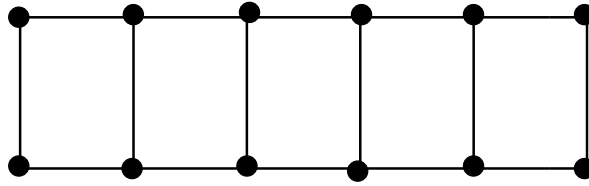


FIGURE 2. 6-ladder

thus vertex 1 is equal to vertex $(n+1)$, vertex $2(n+1)$ is equal to vertex $(n+2)$ in the constructed graph. This new way of viewing an n -necklace will be used to prove the main result of this paper.

3. COMPONENT NUMBER OF A LINK CORRESPONDING TO AN n -LADDER

In this section we give the number of components of a link corresponding to an n -ladder.

The following proposition is stated in [4]

Proposition 3.1. *Let G be a graph. The number of components of a link corresponding to G is the same as the number of components of a link corresponding to G' , where G' , is a graph obtained from G by*

- (a) *contracting a series pair of edges*
- (b) *deleting a pair of parallel edges*
- (c) *contracting a coloop.*

Proposition 3.2. *Let G be an n -ladder, then the number of components of the link corresponding to G is*

- (a) *1 if n is odd*
- (b) *2 if n is even.*

Proof. The proof is by induction on n . It is clear that the link corresponding to a 1-ladder is the unknot, hence has 1 component. The link corresponding to a 2-ladder is 4_1^2 in the table of knots, hence has 2 components. Assume it is true for all $n \leq k-1$. Now consider a k -ladder. By part (a) of Proposition 3.1, start by contracting a series pair of edges at the end of the k -ladder. Thus we get $(k-1)$ -ladder with a pair of parallel edges at the end rung. Then by part (b) of Proposition 3.1 delete the pair of parallel edges, hence creating coloops at the end which we contract by applying part (c) of Proposition 3.1. Thus we end up with an $(k-2)$ -ladder. Hence the number of components of an k -ladder is the same as the number of components of $(k-2)$ -ladder. Thus by induction assumption we get the result. \square

4. CONSTRUCTION OF LINKS

In this section we demonstrate how a corresponding link of a ladder changes to a corresponding link of a necklace when a necklace has been constructed from a ladder. All we need is to show what happens to the construction of a link at the end points when a new graph is formed by merging two edges.

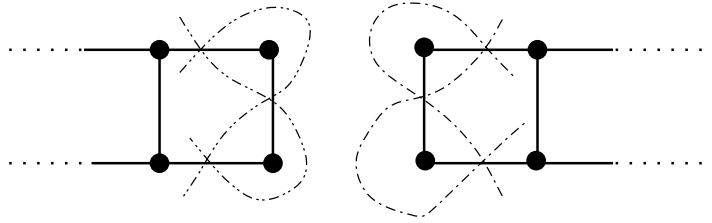


FIGURE 3. End points of a ladder and corresponding link universe before merging

Figure 4 illustrates a simple construction of an n -necklace and its corresponding link from Figure 3. The dashed line indicates where the graph and the corresponding link of a ladder are connected to form a necklace and its corresponding link.

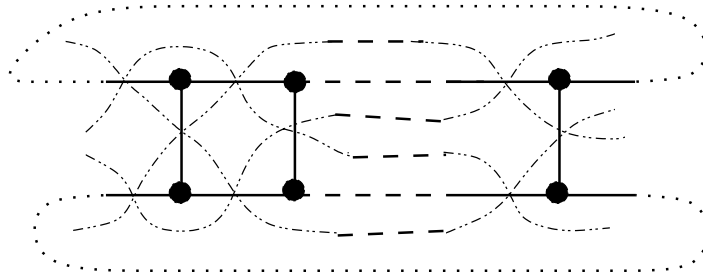


FIGURE 4. Merged end points of ladder and corresponding link universe

To ease notation we shall refer to a ladder-like graph obtained by cutting a pair of edges of a necklace, which are opposite but attached to the same rungs, an n -open-ladder. It is clear from the construction indicated in Figure 4 that if we cut an n -necklace, across the dashed line, we have an n -open-ladder and a braid as shown in figure 5.

Thus it is clear with the labels that in an n -necklace, a string labelled n at one end of the open-ladder is connected to a string with a similar label on the other end. From this it is clear that the number of components of a link of a n -necklace is less or equal to 4.

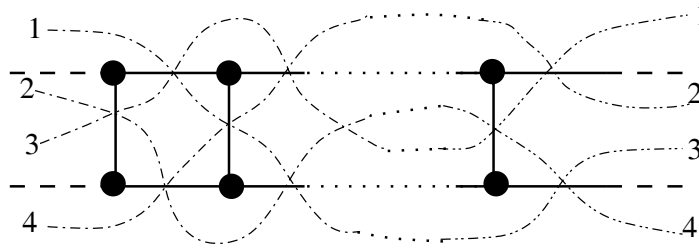


FIGURE 5. Labels for strings at end points of an open-ladder

The following lemma follows immediately. We only verify Lemma 4.1 and similar arguments can be applied to the other lemmas. We will use the notation $A \rightarrow B$ for the braid of an open ladder to mean that a string with label A at one end if followed to the other end it goes to label B .

Lemma 4.1. *Let G be a $4m$ -open-ladder, $m \in \mathbb{Z}$, then the string with label d at one end will have the same label at the other end.*

Proof. It is clear from Figure 6 below that if we follow the strings of the corresponding braid from one end to the other end in a 4-ladder, then $1 \rightarrow 1$, $2 \rightarrow 2$, $3 \rightarrow 3$, and $4 \rightarrow 4$. Thus for any multiple of 4 we have the same result as required.

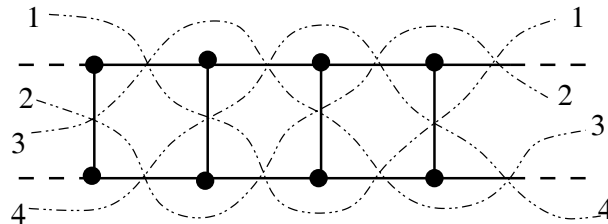


FIGURE 6

□

Lemma 4.2. *Let G be a $(4m + 1)$ -open-ladder, then the string with label (a) $1 \rightarrow 2$ (b) $2 \rightarrow 4$ (c) $3 \rightarrow 1$ (d) $4 \rightarrow 3$.*

Proof. By application of Lemma 4.1 we only need to verify for a 1-ladder. □

Lemma 4.3. *Let G be a $(4m + 2)$ -open-ladder, then the string with label (a) $1 \rightarrow 4$ (b) $2 \rightarrow 3$ (c) $3 \rightarrow 2$ (d) $4 \rightarrow 1$.*

Proof. By application of Lemma 4.1 we only need to verify for a 2-ladder. □

Lemma 4.4. *Let G be a $(4m + 3)$ -open-ladder, then the string with label (a) $1 \rightarrow 3$ (b) $2 \rightarrow 1$ (c) $3 \rightarrow 4$ (d) $4 \rightarrow 2$.*

Proof. By application of Lemma 4.1 we only need to verify for a 3-ladder. \square

5. LINK CORRESPONDING TO AN n -NECKLACE

In this section we state and prove the main result of this paper.

Theorem 5.1. *Let G be an n -necklace, then the number of components of, $L(G)$, the link corresponding to G is*

- (a) 1 if n is odd
- (b) 2 if n is even but not divisible by 4.
- (c) 4 if n is divisible by 4.

Proof. (a) If n is odd then either G is formed by connecting back a $(4m + 1)$ or $(4m + 3)$ open-ladders, $m \in \mathbb{Z}$. We start with, $(4m + 1)$ -open-ladder, by Lemma 4.2 we start by following string $1 \rightarrow 2, 2 \rightarrow 4, 4 \rightarrow 3, 3 \rightarrow 1$. Thus it is clear that $1 \rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow 1$ which forms one cycle, hence the link has one component. Similarly, for a $(4m + 3)$ -open-ladder, by Lemma 4.4, $1 \rightarrow 3 \rightarrow 4 \rightarrow 2 \rightarrow 1$ which forms one cycle, hence the link has one component.

(b) If n is even but not divisible by 4, then $n = 4m + 2$ for $m \in \mathbb{Z}$. Thus by Lemma 4.3, if we connect back a $(4m + 2)$ -open-ladder, we have $1 \rightarrow 4 \rightarrow 1$ and $2 \rightarrow 3 \rightarrow 2$ which forms two cycles. Hence the link has 2 components.

(c) If n is divisible by 4, then $n = 4m$ for $m \in \mathbb{Z}$. Thus by Lemma 4.1, if we connect back a $4m$ -open-ladder, we have $1 \rightarrow 1$, followed by $2 \rightarrow 2$, then $3 \rightarrow 3$ and $4 \rightarrow 4$ which forms four cycles. Hence the corresponding link has 4 components. \square

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