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Kenmotsu and P- Kenmotsu Finsler Structures and Connections on Vector Bundle

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Abstract

The Kenmotsu and para Kenmotsu Finsler structures and their corresponding connections on vector bundle are defined and discussed. A relation between almost contact semi-symmetric metric Finsler connection and almost Kenmotsu Finsler connection has been established. It is found that, if $\overset{o}{\nabla}$ is a torsion free almost Kenmotsu Finsler connection, then the almost contact semi–symmetric metric Finsler connection $\overline{\nabla}$, given by $\overline{\nabla}_X Y = \overset{o}{\nabla}_X Y + \eta(Y)X - G(X,Y)\xi$, satisfies the property of almost Kenmotsu Finsler connection.

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Introduction:

Miron [4] introduced a sophisticated method for the study of Finsler geometry of vector bundles and defined the Finsler Geometry of vector Bundle as the Fisler Geometrical objects theory on the total space of the Vector bundle. Using the concepts of

Miron's theory, Klepp [3] defined 'almost product Finsler structures and connections on vector bundle'. Analogous to this, Sinha & Yadav [8] defined 'almost contact Finsler structures on vector bundle'. They also defined almost contact Finsler connection on vector bundle as semi-symmetric and studied several properties [9].

Sinha and Sai Prasad [7] defined certain Para Kenmosu structures as 'a class of almost para contact metric manifold' in analogy with 'a class of almost contact Riemannian manifolds' defined by Kenmotsu [2]. In addition, Sai Prasad [5] defined 'certain Kenmotsu structures on a differentiable manifold'.

In this paper, the Kenmotsu and para Kenmotsu Finsler structures and their corresponding connections on vector bundle are defined and discussed. A relation between almost contact semi-symmetric metric Finsler connection and almost Kenmotsu Finsler connection has been established. It is found that, if $\overset{o}{\nabla}$ is a torsion free almost Kenmotsu Finsler connection then the almost contact semi–symmetric metric Finsler connection $\overline{\nabla}$, given by $\overline{\nabla}_X Y = \overset{o}{\nabla}_X Y + \eta(Y)X - G(X,Y)\xi$, satisfies the property of almost Kenmotsu Finsler connection.

1. Preliminaries:

Let V(M) = {VM, π, M} be a vector bundle whose total space VM is a (n + m) dimensional c^∞ - manifold and base space M is an n-dimensional c^∞ - manifold.

A non-linear connection N on VM of V(M) is a differentiable distribution

$N: VM \rightarrow T_u(VM) \ni T_u(VM) = N_u \oplus VM_u^v$

where N_u is the horizontal distribution and VM_u^v is the vertical distribution.

Thus for all $X \in T_u(VM)$ can be decomposed as

$X = X^H + X^V$; where $X^H \in N_u$ and $X^V \in VM_u^v$.

A Finsler connection F $\Gamma = \nabla$ on the total space VM is a linear connection with the property that the horizontal linear space N_u and the vertical linear space VM_u^v are parallel with respect to ∇.

Therefore, a linear connection ∇ on VM is a Finsler connection if and only if $\left(\nabla_X Y^H\right)^V = 0$ and $\left(\nabla_X Y^V\right)^H = 0$ for all X , Y $\in T_u(VM)$.

A Finsler connection ∇ on VM is characterized by the horizontal part ∇^H and the vertical part ∇^V [4].

A linear connection ∇ on VM with properties:

1. ∇ is an almost contact connection on VM
2. ∇ is a Finsler connection relative to the distributions N and VM^v
   i.e., $\left(\nabla_X Y^H\right)^V = 0$, $\left(\nabla_X Y^V\right)^H = 0$

is called an almost contact Finsler connection on VM [3].

A Finsler connection ∇ on VM is said to be semi-symmetric [4] if its torsion tensor T satisfies:

$$T(X, Y) = \nabla_x Y - \nabla_Y X - [X, Y]$$
$$= X\eta(Y) - Y\eta(X)$$

which give,

$$[T(X^H, Y^H)]^H = X^H\eta^H(Y^H) - Y^H\eta^H(X^H)$$

$$[T(X^V, Y^V)]^V = X^V\eta^V(Y^V) - Y^V\eta^V(X^V);\ \forall X, Y \in T_u(VM). \qquad (2.1)$$

3. Almost Kenmotsu Finsler Structures on Vector Bundle:

Let (ϕ, η, ξ, G) be an almost contact metrical Finsler structure on VM where ϕ is a Finsler tensor field of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, ξ is a vector field, η is a 1-form and G is the Finsler metric structure [8].

Now we define,

$$\Omega (X , Y) = G (\phi X , Y) \text{ i.e.,}$$

$$\Omega (X^H, Y^H) = G^H (\phi X, Y); \Omega (X^v, Y^v) = G^v (\phi X, Y) \qquad (3.1)$$

and call it the fundamental 2-form.

Proposition (3.1): The fundamental 2 - form defined by (3.1) satisfies

$$\Omega (\phi X^H, \phi Y^H) = \Omega (X^H, Y^H) ; \Omega (\phi X^v , \phi Y^v) = \Omega (X^v, Y^v)$$

$$\Omega (X^H, Y^H) = -\Omega (Y^H , X^H) ; \Omega (X^v , Y^v) = -\Omega (Y^v , X^v); \quad (3.2)$$

$\forall X, Y \in T_u (VM)$.

Definition (3.1): Let ∇ be a Finsler connection on VM and η be the 1-form which satisfies $d\eta(X , Y) = 0$, then there exists a function f on VM such that $\eta = df$, i.e.,

$$(\nabla_X{}^H \eta)(Y^H) - (\nabla_Y{}^H \eta)(X^H) + \eta (T(X^H, Y^H)) = 0,$$

$$(\nabla_X{}^H \eta)(Y^V) - (\nabla_Y{}^V \eta)(X^H) + \eta (T(X^H, Y^V)) = 0, \qquad (3.3)$$

$$(\nabla_X{}^V \eta)(Y^V) - (\nabla_Y{}^V \eta)(X^V) + \eta (T(X^V, Y^V)) = 0.$$

Then the almost contact metrical Finsler structure is called an almost Kenmotsu Finsler structure (or) contact metric Finsler structure and the Finsler connection ∇ staisfying(3.3) is called an almost Kenmotsu Finsler connection on VM.

Theorem(3.1): If the almost Kenmotsu Finsler connection ∇ on VM is torsion free, then

$$(\nabla_X{}^H \eta)(Y^H) - (\nabla_Y{}^H \eta)(X^H) = 0$$

$$(\nabla_X{}^H \eta)(Y^V) - (\nabla_Y{}^V \eta)(X^H) = 0 \qquad (3.4)$$

$$(\nabla_X{}^V \eta)(Y^V) - (\nabla_Y{}^V \eta)(X^V) = 0; \quad \forall X, Y \in T_u (VM).$$

Proof: If the Finsler connection ∇ on VM is without torsion[4], then

$$T(X^H, Y^H) = 0; \qquad T(X^H, Y^V) = 0; \qquad T(X^V, Y^V) = 0. \qquad (3.5)$$

Hence from (3.5) and (3.3) we have (3.4).

Definition (3.2): An almost Kenmotsu Finsler structure on VM is an S-contact metric Finsler structure if the 1-form η satisfies:

$$(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) = 2G(\phi X , \phi Y);$$

that is,

$$(\nabla_X{}^H \eta)(Y^H) + (\nabla_Y{}^H \eta)(X^H) = 2G^H(\phi X , \phi Y)$$

$$(\nabla_X{}^V \eta)(Y^V) + (\nabla_Y{}^V \eta)(X^V) = 2G^V(\phi X , \phi Y);\ \forall X, Y \in T_u(VM) \qquad (3.6)$$

and the Finsler connection ∇ on VM, which is torsion free, is called an S-contact metric Finsler connection.

Theorem(3.2): Let ∇ be the torsion free Finsler connection together with an S-contact metric Finsler structure on VM and Ω be the fundamental 2-form then $\forall X, Y \in T_u(VM)$;

$$(\nabla_X{}^H \eta)(Y^H) = (\nabla_Y{}^H \eta)(X^H) = -\Omega(\phi X^H, Y^H)$$

$$(\nabla_X{}^V \eta)(Y^V) = (\nabla_Y{}^V \eta)(X^V) = -\Omega(\phi X^V, Y^V) \qquad (3.7)$$

Proof : From (3.4) and (3.6) we have (3.7).

Also, from (3.1) and (3.7), we have

$$\nabla x^H \xi^H = X^H - \eta^H(X^H)\,\xi^H$$

$$\nabla x^V \xi^V = X^V - \eta^V(X^V)\,\xi^V. \qquad (3.8)$$

It may be noted that the vector bundle may called Kenmotsu vector bundle *provided Kenmotsu Finsler structures are defined on its total space.*

4. P-Kenmotsu and SP-Kenmotsu Finsler Structures on Vector Bundle:

Let (ϕ, η, ξ, G) be an almost para contact metrical Finsler structure on VM where ϕ is a Finsler tensor field of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, ξ is a vector field, η is a 1-form and G is the Finsler metric structure [6].

Now we define Ω as

$$\Omega (X , Y) = G (\phi X , Y) \text{ i.e.,}$$

$$\Omega (X^H, Y^H) = G^H (\phi X, Y) ; \Omega (X^v, Y^v) = G^v (\phi X, Y) \qquad (4.1)$$

and call it the covariant tensor.

Proposition (4.1): The covariant tensor defined by (4.1) satisfies

$$\text{(i) } \Omega (\phi X^H, \phi Y^H) = \Omega (X^H, Y^H) ; \Omega (\phi X^v , \phi Y^v) = \Omega (X^v , Y^v)$$

$$\text{(ii) } \Omega (X^H, Y^H) = \Omega (Y^H, X^H) ; \Omega (X^v , Y^v) = \Omega (Y^v, X^v); \qquad (4.2)$$

$$\forall X, Y \in T_u (VM).$$

Definition(4.1): The vector bundle VM with Riemannian Finsler metric G admitting a Finsler tensor field ϕ of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, a vector field ξ and a 1-form η satisfying

$$\text{(i) } (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0$$

$$\text{(ii) } (\nabla_X \nabla_Y \eta)(Z) = [-G (X , Z) + \eta(X) \eta(Z)] \eta(Y) + [-G (X , Y) + \eta(X) \eta(Y)] \eta(Z)$$

$$\text{(iii) } \eta(X) = G (X , \xi)$$

$$\text{(iv) } \nabla_X \xi = \phi^2 X = \overline{\overline{X}} = X - \eta(X) \xi \qquad (4.3)$$

is said to possess the P-Kenmotsu Finsler structure. i.e., for (4.3.i), we obtain

$$(\nabla_{X^H} \eta) (Y^H) - (\nabla_{Y^H} \eta) (X^H) = 0$$

$$(\nabla_{X^H} \eta) (Y^V) - (\nabla_{Y^V} \eta) (X^H) = 0$$

$$(\nabla_{X^V} \eta) (Y^V) - (\nabla_{Y^V} \eta) (X^V) = 0; \qquad \forall X, Y \in T_u (VM).$$

Similar results can be obtained for the remaining equations of (4.3). The torsion free Finsler connection ∇ satisfying (4.3) is called P-Kenmotsu Finsler connection on VM.

Definition (4.2): The vector bundle VM with Riemannian Finsler metric G admitting a Finsler tensor field ϕ of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, a vector field ξ and a 1-form η satisfying

$$\text{(i) } (\nabla_X \eta) (Y) = G (X , Y) - \eta(X) \eta(Y)$$

$$\text{(ii) } \eta(X) = G(X, \xi) \text{ and } (\nabla_X \eta)(Y) = \Omega(\overline{X}, Y) \tag{4.4}$$

is said to possess the SP-Kenmotsu Finsler structure i.e., for (4.4.i), we obtain

$$(\nabla_{X^H}\eta)(Y^H) = G^H(X, Y) - \eta^H(X)\,\eta^H(Y)$$
$$(\nabla_{X^V}\eta)(Y^V) = G^V(X, Y) - \eta^V(X)\,\eta^V(Y); \quad \forall\, X, Y \in T_u(VM)$$

Similar results can be obtained for (4.4.ii). The torsion free Finsler connection ∇ satisfying (4.4) is called an SP-Kenmotsu Finsler connection on VM.
Therefore, for an SP-Kenmotsu Finsler structure on VM, from (4.4) , we have

$$\begin{aligned}\Omega(\overline{X}, Y) &= G(X, Y) - \eta(X)\,\eta(Y) \text{ i.e.,}\\ \Omega(\overline{X}^H, Y^H) &= G^H(X, Y) - \eta^H(X)\,\eta^H(Y)\\ \Omega(\overline{X}^V, Y^V) &= G^V(X, Y) - \eta^V(X)\,\eta^V(Y)\ ; \quad \forall\, X, Y \in T_u(VM).\end{aligned} \tag{4.5}$$

It may be noted that the vector bundle may called ***P-Kenmotsu*** *or* ***SP-Kenmotsu*** *vector bundle* provided *respective Finsler structures are defined on its total space.*

5. Almost Contact Semi-Symmetric Metric Finsler Connection:

An almost contact semi-symmetric Finsler connection $\overline{\nabla}$ on VM is said to be an almost contact semi-symmetric metric Finsler connection if and only if

$$\overline{\nabla}_X G^H = 0 \text{ and } \overline{\nabla}_X G^V = 0.$$

If $\overset{o}{\nabla}$ be the almost contact Finsler connection on VM which is torsion free then an almost contact semi-symmetric metric Finsler connection $\overline{\nabla}$ is given by

$$\overline{\nabla}_X Y = \overset{o}{\nabla}_X Y + H(X, Y)\,; \forall\, X, Y \in T_u(VM) \tag{5.1}$$

where

$$H(X, Y) = \tfrac{1}{2}\left\{ T(X, Y) + P(X, Y) + P(Y, X) \right\} \tag{5.2}$$

$$P(X, Y) = \eta(X)\,Y - G(X, Y)\,\xi. \tag{5.3}$$

From (2.1), (5.2) and (5.3) we have

$$H(X, Y) = \eta(Y)X - G(X, Y)\xi. \tag{5.4}$$

Theorem (5.1): Let $\overline{\nabla}$ be an almost contact semi-symmetric metric Finsler connection and $\overset{o}{\nabla}$ be an S-contact metric Finsler connection which is torsion free, then $\forall$ X, Y $\in T_u(VM)$;

$$\begin{aligned}\left(\overline{\nabla}^{H}_{\phi X}\eta\right)\left(\phi Y^{H}\right) + \left(\overline{\nabla}^{H}_{\phi Y}\eta\right)\left(\phi X^{H}\right) &= 0\\ \left(\overline{\nabla}^{V}_{\phi X}\eta\right)\left(\phi Y^{V}\right) + \left(\overline{\nabla}^{V}_{\phi Y}\eta\right)\left(\phi X^{V}\right) &= 0\end{aligned} \tag{5.5}$$

Proof : From (5.1) and (5.4), we have

$$\overline{\nabla}_X Y = \overset{o}{\nabla}_X Y + \eta(Y)X - G(X, Y)\xi \tag{5.6}$$

From the above, we get

$$(\overline{\nabla}_X \eta)(Y) = (\overset{o}{\nabla}_X \eta)(Y) - G(\phi X, \phi Y)$$

Replacing X by ϕX and Y by ϕY, we get

$$(\overline{\nabla}_{\phi X}\eta)(\phi Y) = (\overset{o}{\nabla}_{\phi X}\eta)(\phi Y) - G(\phi X, \phi Y).$$

Therefore,

$$\begin{aligned}&(\overline{\nabla}_{\phi X}\eta)(\phi Y) + (\overline{\nabla}_{\phi Y}\eta)(\phi X)\\ &\quad= (\overset{o}{\nabla}_{\phi X}\eta)(\phi Y) + (\overset{o}{\nabla}_{\phi Y}\eta)(\phi X) - 2G(\phi X, \phi Y)\\ &\quad= 0,\end{aligned}$$

which gives (5.5). [Since $\overset{o}{\nabla}$ is an S-contact metric Finsler connection, $(\overset{o}{\nabla}_{\phi X}\eta)(\phi Y) + (\overset{o}{\nabla}_{\phi Y}\eta)(\phi X) = 2G(\phi X, \phi Y)$].

Theorem(5.2): Let $\overline{\nabla}$ be an almost contact semi-symmetric metric Finsler connection and $\overset{o}{\nabla}$ be an almost Kenmotsu Finsler connection which is torsion free, then

$$\begin{aligned}\left(\overline{\nabla}^{H}_{X}\eta\right)\left(Y^{H}\right) - \left(\overline{\nabla}^{H}_{Y}\eta\right)\left(X^{H}\right) &= 0\\ \left(\overline{\nabla}^{V}_{X}\eta\right)\left(Y^{V}\right) - \left(\overline{\nabla}^{V}_{Y}\eta\right)\left(X^{V}\right) &= 0\end{aligned} \qquad \forall\, X, Y \in T_u(VM). \tag{5.7}$$

Proof : From (5.6) we have

$$(\overline{\nabla}_X \eta)(Y) = (\overset{o}{\nabla}_X \eta)(Y) - G(\phi X, \phi Y)$$

and $$(\overline{\nabla}_Y \eta)(X) = (\overset{o}{\nabla}_Y \eta)(X) - G(\phi Y, \phi X)$$

Subtracting these two and using (3.4), we get

$$(\overline{\nabla}_X \eta)(Y) - (\overline{\nabla}_Y \eta)(X) = 0.$$

Since $\overline{\nabla}$ is a Finsler connection, we have

$$\left(\overline{\nabla}^H_X \eta\right)\left(Y^H\right) - \left(\overline{\nabla}^H_Y \eta\right)\left(X^H\right) + \left(\overline{\nabla}^V_X \eta\right)\left(Y^V\right) - \left(\overline{\nabla}^V_Y \eta\right)\left(X^V\right) = 0$$

which gives (5.7).

Theorem (5.3): Let $\overline{\nabla}$ be an almost contact semi-symmetric metric Finsler connection and $\overset{o}{\nabla}$ be an S-contact metric Finsler connection which is torsion free, then

$$\begin{aligned} \left(\overline{\nabla}^H_X \eta\right)\left(Y^H\right) + \left(\overline{\nabla}^H_Y \eta\right)\left(X^H\right) &= 0 \\ \left(\overline{\nabla}^V_X \eta\right)\left(Y^V\right) + \left(\overline{\nabla}^V_Y \eta\right)\left(X^V\right) &= 0 \end{aligned} \qquad \forall\, X, Y \in T_u(VM). \tag{5.8}$$

Proof : From (5.6) we have

$$(\overline{\nabla}_x \eta)(Y) = (\overset{o}{\nabla}_x \eta)(Y) - G(\phi X, \phi Y).$$

Therefore,

$$\begin{aligned} &(\overline{\nabla}_x \eta)(Y) + (\overline{\nabla}_Y \eta)(X) \\ &\quad = (\overset{o}{\nabla}_x \eta)(Y) + (\overset{o}{\nabla}_Y \eta)(X) - 2G(\phi X, \phi Y) \\ &\quad = 0, \text{ from (3.6).} \end{aligned}$$

Since $\overline{\nabla}$ is a Finsler connection, we have

$$\left(\overline{\nabla}^H_X \eta\right)\left(Y^H\right) + \left(\overline{\nabla}^V_X \eta\right)\left(Y^v\right) + \left(\overline{\nabla}^H_Y \eta\right)\left(X^H\right) + \left(\overline{\nabla}^V_Y \eta\right)\left(X^V\right) = 0$$

which gives (5.8).

Conclusion:

It is found that, if $\overset{o}{\nabla}$ is a torsion free almost Kenmotsu Finsler connection, then the almost contact semi–symmetric metric Finsler connection $\overline{\nabla}$, given by $\overline{\nabla}_X Y = \overset{o}{\nabla}_X Y + \eta(Y)X - G(X,Y)\xi$, satisfies the property of almost Kenmotsu Finsler connection.

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