International Mathematical Forum, 3, 2008, no. 19, 937 - 944

An Identity on θ-Centralizers of Semiprime Rings

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Abstract

The purpose of this note is to prove the following result. Let R be a 2-torsion free semiprime ring and let $T : R \longrightarrow R$ be an additive mapping such that $2T(xyx) = T(x)\theta(yx) + \theta(xy)T(x)$ holds for all $x, y \in R$, where θ is a homomorphism from R onto R. Then T is a $\theta-$centralizer of R.

Mathematics Subject Classification: 16N60, 39B05

Keywords: Prime ring, Semiprime ring, Derivation, Jordan derivation, Jordan triple derivation, Left (right) centralizer, Left (right) Jordan centralizer, Centralizer, Left (right) $\theta-$centralizer, Left (right) Jordan $\theta-$centralizer, $\theta-$centralizer

1 Introduction

This note is motivated by the work of Vukman and Kosi-Ulbl [7]. Throughout this note, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where n is an integer, in case $nx = 0$, $x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use basic commutator identities $[x, yz] = [x, y]z + y[x, z]$ and $[xz, y] = [x, y]z + x[z, y]$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \longrightarrow R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$, and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. A derivation D is inner in case there exists $a \in R$ such that $D(x) = [a, x]$. Every derivation is a Jordan derivation. The inverse is in general not true. A classical result of Herstein ([6]) asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in ([2]). Cusack ([5]) has generalized Herstein's result to 2-torsion free semiprime rings (see also [3] for an alternative proof). An additive mapping $T : R \longrightarrow R$ is called a left (right) centralizer in case $T(xy) = T(x)y \quad (T(xy) = xT(y))$ holds for all $x, y \in R$. We follow Zalar [10] and call T a centralizer in case T is both a left and a right centralizer. If $a \in R$ then $L_a(x) = ax$ is a left centralizer and $R_a(x) = xa$ is a right centralizer. An additive mapping $T : R \longrightarrow R$ is called a left (right) Jordan centralizer in case $T(x^2) = T(x)x$ $(T(x^2) = xT(x))$. Following ideas from [3], Zalar ([10]) has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. Also, Vukman ([8]) proved that if $T : R \longrightarrow R$ is an additive mapping such that $2T(x^2) = T(x)x + xT(x)$ holds for all $x \in R$, then T is a centralizer. Also, Vukman ([9]) proved that if R is a 2-torsion free semiprime ring and $T : R \longrightarrow R$ is an additive mapping such that $T(xyx) = xT(y)x$ holds for all $x, y \in R$, then T is a centralizer. In ([7]) Vukman and Irena proved that if R is a 2-torsion free semiprime ring and $T : R \longrightarrow R$ is an additive mapping such that $2T(xyx) = T(x)yx + xyT(x)$ holds for all $x, y \in R$, then T is a centralizer. An additive mapping $D : R \longrightarrow R$, where R is an arbitrary ring, is a Jordan triple derivation in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all $x, y \in R$. One can easily prove that any Jordan triple derivation is a triple derivation (see[2]). Bresar ([4]) has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a triple derivation. In [1], Albas has introduced the notation of θ-centralizer and Jordan θ-centralizer, which is a generalization of the definition of centralizer and Jordan centralizer, and he proved, on a 2-torsion free semiprime ring, with some condition that every Jordan θ-centralizer is a θ-centralizer. In this note, Vukman and Irena's identity ([7]) will be proved using Albas's definition.

Definition 1.1. *[1] An additive mapping* $T : R \longrightarrow R$ *is called a left (right)*

$\theta-$centralizer associated with a function $\theta : R \longrightarrow R$ if for all $x, y \in R$,

$$T(xy) = T(x)\theta(y) \quad (T(xy) = \theta(x)T(y)).$$

And T is called a left (right) Jordan $\theta-$centralizer if for all $x, y \in R$,

$$T(x^2) = T(x)\theta(x) \quad (T(x^2) = \theta(x)T(x)).$$

Remark. Clearly every centralizer is a special case of a θ-centralizer with $\theta = I_R$.

If $T : R \longrightarrow R$ is a θ-centralizer associated with a function $\theta : R \longrightarrow R$, where R is an arbitrary ring, then T satisfies the relation

$$2T(xyx) = T(x)\theta(yx) + \theta(xy)T(x) \quad \forall\, x, y \in R. \tag{1}$$

It seems natural to ask, as Vukman and Irena have done for the centralizer case, whether the converse is true. More precisely, we are asking whether an additive mapping T on a ring R satisfying relation (1) is a θ-centralizer for all $x, y \in R$. It is the aim in this paper to prove that the answer is affirmative when R is a 2-torsion free semiprime ring and θ is a surjective homomorphism.

2 The Main Result

We now give the main result of this paper.

Theorem 2.1. *Let R be a 2-torsion free semiprime ring and let $T : R \longrightarrow R$ be an additive mapping. Suppose that $2T(xyx) = T(x)\theta(yx) + \theta(xy)T(x)$ holds for all pairs $x, y \in R$ and $\theta(Z(R)) = Z(R)$ where θ be a nonzero surjective endomorphism on R. Then T is a θ-centralizer.*

For the proof of theorem (2.1) the following lemma will be needed, this lemma can be founded in ([9], Lemma 1).

Lemma 2.2. *Let R be a semiprime ring. Suppose that the identity $axb + bxc = 0$ holds for all $x \in R$ and some $a, b, c \in R$. In this case $(a + c)xb = 0$ is satisfied for all $x \in R$.*

Proof. of theorem (2.1): Putting $x + z$ for x in (1), we obtain

$$2T(xyz + zyx) = T(x)\theta(yz) + T(z)\theta(yx) + \theta(zy)T(x) + \theta(xy)T(z), \quad \forall x, y, z \in R. \tag{2}$$

Letting $z = x^2$ we have

$$2T(xyx^2 + x^2yx) = T(x)\theta(yx^2) + T(x^2)\theta(yx) + \theta(x^2y)T(x) + \theta(xy)T(x^2), \qquad \forall x, y, z \in R, \tag{3}$$

Replacing y by $xy + yx$ in (1), we obtain

$$2T(xyx^2 + x^2yx) = T(x)\theta(xyx) + T(x)\theta(yx^2) + \theta(x^2y)T(x) + \theta(xyx)T(x), \qquad \forall x, y, z \in R. \tag{4}$$

Subtracting (4) from (3), gives

$$(T(x^2) - T(x)\theta(x))\theta(yx) + \theta(xy)(T(x^2) - \theta(x)T(x)) = 0, \qquad \forall x, y \in R.$$

Taking $a = T(x^2) - T(x)\theta(x)$, $b = \theta(x)$, $z = \theta(y)$ and $c = T(x^2) - \theta(x)T(x)$ the above relation can be rewritten in the form

$$azb + bzc = 0, \quad \forall z \in R.$$

Using Lemma (2.2) we get

$$(2T(x^2) - T(x)\theta(x) - \theta(x)T(x))\theta(y)\theta(x) = 0, \qquad \forall x, y \in R.$$

If we take $A(x) = 2T(x^2) - T(x)\theta(x) - \theta(x)T(x)$ then the above relation can be rewritten as follows

$$A(x)\theta(yx) = 0, \qquad \forall x, y \in R. \tag{5}$$

Replacing $\theta(y)$ by $\theta(xy)A(x)$ in (5), we obtain

$$A(x)\theta(x)\theta(y)A(x)\theta(x) = 0, \qquad \forall x, y \in R,$$

By the surjectivity of θ and the semiprimeness of R it follows that

$$A(x)\theta(x) = 0, \qquad \forall x \in R. \tag{6}$$

Similarly, if we multiplying (5) from the left by $\theta(x)$ and from the right side by $A(x)$, we obtain

$$\theta(x)A(x)\theta(y)\theta(x)A(x) = 0, \qquad \forall x, y \in R,$$

This gives

$$\theta(x)A(x) = 0, \qquad \forall x \in R. \tag{7}$$

Replacing x by $x + y$ in (6) gives

$$A(x)\theta(y) + A(y)\theta(x) + B(x, y)\theta(x) + B(x, y)\theta(y) = 0, \qquad \forall x, y \in R,$$

where $B(x,y)$ stands for $2T(xy+yx)-T(x)\theta(y)-T(y)\theta(x)-\theta(x)T(y)-\theta(y)T(x)$.
Replacing x by $-x$ in the above relation and comparing the new relation with the old one we arrive at

$$A(x)\theta(y)+B(x,y)\theta(x)=0, \qquad \forall x,y\in R. \tag{8}$$

Right multiplication of the above relation by $A(x)$ along with (7) gives

$$A(x)\theta(y)A(x)=0, \qquad \forall x,y\in R,$$

By the surjectivity of θ and the semiprimeness of R it follows that

$$A(x)=0, \qquad \forall x\in R.$$

We now have

$$2T(x^2)=T(x)\theta(x)+\theta(x)T(x), \qquad \forall x\in R. \tag{9}$$

We intend to prove that

$$[T(x),\theta(x)]=0, \qquad \forall x\in R. \tag{10}$$

Replacing x by $x+y$ in relation (9) gives

$$2T(xy+yx)=T(x)\theta(y)+T(y)\theta(x)+\theta(x)T(y)+\theta(y)T(x), \qquad \forall x,y\in R. \tag{11}$$

Replacing y with $2xyx$ in (11), using the assumption of the theorem, we obtain

$$4T(x^2yx+xyx^2)=$$
$$2T(x)\theta(xyx)+2T(xyx)\theta(x)+2\theta(x)T(xyx)+2\theta(xyx)T(x)=2T(x)\theta(xyx)+$$
$$T(x)\theta(yx^2)+\theta(xy)T(x)\theta(x)+\theta(x)T(x)\theta(yx)+\theta(x^2y)T(x)+2\theta(xyx)T(x).$$

Thus we have

$$\begin{aligned}4T(x^2yx+xyx^2)&=2T(x)\theta(xyx)+T(x)\theta(yx^2)+\theta(xy)T(x)\theta(x)\\&+\theta(x)T(x)\theta(yx)+\theta(x^2y)T(x)+2\theta(xyx)T(x), \qquad \forall x,y\in R.\end{aligned} \tag{12}$$

By comparing (4) and (12), we arrive at

$$\begin{aligned}T(x)\theta(yx^2)+\theta(x^2y)T(x)-\theta(xy)T(x)\theta(x)-\theta(x)T(x)\theta(yx)=0,\\ \forall x,y\in R.\end{aligned} \tag{13}$$

Replacing y by yx in the above relation, we obtain

$$\begin{aligned}T(x)\theta(yx^3)+\theta(x^2yx)T(x)-\theta(xyx)T(x)\theta(x)-\theta(x)T(x)\theta(yx^2)=0,\\ \forall x,y\in R.\end{aligned} \tag{14}$$

Right multiplication of (13) by $\theta(x)$ gives

$$T(x)\theta(yx^3) + \theta(x^2y)T(x)\theta(x) - \theta(xy)T(x)\theta(x^2) - \theta(x)T(x)\theta(yx^2) = 0, \quad \forall x, y \in R. \tag{15}$$

Subtracting (14) from (15), we arrive at

$$\theta(x^2)\theta(y)[T(x), \theta(x)] - \theta(xy)[T(x), \theta(x)]\theta(x) = 0, \qquad \forall x, y \in R. \tag{16}$$

The substitution $T(x)\theta(y)$ for $\theta(y)$ in (16) leads to

$$\theta(x^2)T(x)\theta(y)[T(x), \theta(x)] - \theta(x)T(x)\theta(y)[T(x), \theta(x)]\theta(x) = 0, \quad \forall x, y \in R. \tag{17}$$

Left multiplication of (16) by $T(x)$ gives

$$T(x)\theta(x^2)\theta(y)[T(x), \theta(x)] - T(x)\theta(x)\theta(y)[T(x), \theta(x)]\theta(x) = 0, \quad \forall x, y \in R. \tag{18}$$

Subtracting (17) from (18), we arrive at

$[T(x), \theta(x^2)]\theta(y)[T(x), \theta(x)] - [T(x), \theta(x)]\theta(y)[T(x), \theta(x)]\theta(x) = 0,\ \forall x, y \in R.$

In the above relation let

$$a = [T(x), \theta(x^2)], b = [T(x), \theta(x)], c = -[T(x), \theta(x)]\theta(x) \text{ and } z = \theta(y)$$

With these substitutions we have

$$azb + bzc = 0.$$

We apply Lemma (2.2) to the above relation. This gives us

$$([T(x), \theta(x^2)] - [T(x), \theta(x)]\theta(x))\theta(y)[T(x), \theta(x)] = 0, \qquad \forall x, y \in R.$$

This is equivalent to

$$\theta(x)[T(x), \theta(x)]\theta(y)[T(x), \theta(x)] = 0, \qquad \forall x, y \in R.$$

Replacing $\theta(y)$ by $\theta(yx)$ in the above relation, we obtain

$$\theta(x)[T(x), \theta(x)]\theta(y)\theta(x)[T(x), \theta(x)] = 0, \qquad \forall x, y \in R,$$

By the surjectivity of θ and the semiprimeness of R it follows that

$$\theta(x)[T(x), \theta(x)] = 0, \qquad \forall x \in R. \tag{19}$$

Replacing y for xy in relation (13) we obtain

$$T(x)\theta(xyx^2) + \theta(x^3y)T(x) - \theta(x^2y)T(x)\theta(x) - \theta(x)T(x)\theta(xyx) = 0, \quad \forall x, y \in R. \tag{20}$$

Left multiplication of (13) by $\theta(x)$ gives

$$\theta(x)T(x)\theta(yx^2) + \theta(x^3y)T(x) - \theta(x^2y)T(x)\theta(x) - \theta(x^2)T(x)\theta(yx) = 0, \quad \forall x, y \in R. \tag{21}$$

Subtracting (21) from (20), we arrive at

$$[T(x), \theta(x)]\theta(y)\theta(x^2) - \theta(x)[T(x), \theta(x)]\theta(y)\theta(x) = 0, \qquad \forall x, y \in R.$$

Using relation (19) in the above relation we get

$$[T(x), \theta(x)]\theta(y)\theta(x^2) = 0, \qquad \forall x, y \in R. \tag{22}$$

The substitution $\theta(y)T(x)$ for $\theta(y)$ in (22) leads to

$$[T(x), \theta(x)]\theta(y)T(x)\theta(x^2) = 0, \qquad \forall x, y \in R. \tag{23}$$

Right multiplication of (22) by $T(x)$ gives

$$[T(x), \theta(x)]\theta(y)\theta(x^2)T(x) = 0, \qquad \forall x, y \in R. \tag{24}$$

Subtracting (24) from (23), we obtain

$$[T(x), \theta(x)]\theta(y)[T(x), \theta(x^2)] = 0, \qquad \forall x, y \in R.$$

The above relation can be rewritten using relation (19) as

$$[T(x), \theta(x)]\theta(y)[T(x), \theta(x)]\theta(x) = 0, \qquad \forall x, y \in R.$$

Replacing y for xy in the above gives

$$[T(x), \theta(x)]\theta(x)\theta(y)[T(x), \theta(x)]\theta(x) = 0, \qquad \forall x, y \in R.$$

By the surjectivity of θ and the semiprimeness of R it follows that

$$[T(x), \theta(x)]\theta(x) = 0, \qquad \forall x \in R. \tag{25}$$

Replacing x by $x + y$ in relation (19) and then using (19) gives

$$\theta(x)[T(x), \theta(y)] + \theta(x)[T(y), \theta(x)] + \theta(x)[T(y), \theta(y)] + \theta(y)[T(x), \theta(x)] + \theta(y)[T(x), \theta(y)] + \theta(y)[T(y), \theta(x)] = 0, \quad \forall\, x, y \in R.$$

Replacing x by $-x$ in the above relation and comparing the relation so obtained with the above relation we get:

$$\theta(x)[T(x), \theta(y)] + \theta(x)[T(y), \theta(x)] + \theta(y)[T(x), \theta(x)] = 0, \quad \forall\, x, y \in R. \tag{26}$$

We left multiple the above relation by $[T(x), \theta(x)]$ and then use (25) to get

$$[T(x), \theta(x)]\theta(y)[T(x), \theta(x)] = 0, \quad \forall\, x, y \in R.$$

By the surjectivity of θ and the semiprimeness of R we get,

$$[T(x), \theta(x)] = 0, \quad \forall\, x \in R.$$

Relation (10) follows. Combining (9) and (10) we get

$$T(x^2) = T(x)\theta(x), \quad \forall\, x \in R,$$

and

$$T(x^2) = \theta(x)T(x), \quad \forall\, x \in R,$$

This means that T is a left and also a right Jordan $\theta-$centralizer. By Theorem (2) in [1] it follows that T is a left and also right $\theta-$centralizer, which completes the proof. □

Corollary 2.3. *([7]Theorem 1) Let R be a 2-torsion free semiprime ring and let $T : R \longrightarrow R$ be an additive mapping. Suppose that $2T(xyx) = T(x)yx + xyT(x)$ holds for all pairs $x, y \in R$. Then T is a centralizer.*

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Received: October 6, 2007