Intuitionistic \((T, S)\)-Normed Fuzzy Closed Ideals of BCH-Algebra

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Abstract

We consider the generalization of the notion of fuzzy subalgebras and closed ideal in BCH-algebras. In this paper, using \(t\)-norm \(T\) and \(s\)-norm \(S\), we introduce the notion of intuitionistic \((T, S)\)-normed fuzzy subalgebra and intuitionistic \((T, S)\)-normed fuzzy closed ideal in BCH-algebras, and some related properties are investigated.

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1 Introduction

In 1966, Y. Imai and K. Iséki ([10]) and K. Iséki ([11]) introduced two classes of abstract algebras: \(BCK\)-algebras and \(BCI\)-algebras. It is known that the class of \(BCK\)-algebras is a proper subclass of the class of \(BCI\)-algebras. In 1983, Q. P. Hu and X. Li ([8, 9]) introduced a wide class of abstract algebras: \(BCH\)-algebras. They have shown that the class of \(BCI\)-algebras is a proper subclass of the class of \(BCH\)-algebras. They have studied some properties of these algebras. Certain other properties have been studied by B. Ahmad ([2]), M. A. Chaudhry ([5]), W. A. Dudek and J. Thomys([7]). After the introduction of the concept of fuzzy sets by Zadeh [18], several researches were conducted on the generalization of the notion of fuzzy sets. The idea of “intuitionistic fuzzy set” was first published by Atanassov [3, 4], as a generalization of the notion of fuzzy set. In this paper, using \(t\)-norm \(T\) and \(s\)-norm \(S\), we introduce the notion of intuitionistic \((T, S)\)-normed fuzzy subalgebra and intuitionistic \((T, S)\)-normed fuzzy closed ideal in BCH-algebras, and some related properties are investigated.
2 Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

By a \textit{BCH-algebra} we mean an algebra \((X, \ast, 0)\) of type \((2, 0)\) satisfying the following axioms:

\begin{align*}
\text{(H1)} & \quad x \ast x = 0, \\
\text{(H2)} & \quad x \ast y = 0 \text{ and } y \ast x = 0 \text{ imply } x = y, \\
\text{(H3)} & \quad (x \ast y) \ast z = (x \ast z) \ast y, \text{ for every } x, y, z \in X.
\end{align*}

In a BCH-algebra \(X\), the following statements hold:

\begin{align*}
\text{(P1)} & \quad x \ast 0 = x. \\
\text{(P2)} & \quad x \ast 0 = 0 \text{ implies } x = 0. \\
\text{(P3)} & \quad x \ast (x \ast y) = (0 \ast x) \ast (0 \ast y).
\end{align*}

A non-empty subset \(A\) of a BCH-algebra \(X\) is called a \textit{subalgebra} of \(X\) if \(x \ast y \in A\) whenever \(x, y \in A\). A nonempty subset \(A\) of a BCH-algebra \(X\) is called a \textit{closed ideal} of \(X\) if

\begin{enumerate}
\item[(i)] \(0 \ast x \in A\) for all \(x \in A\),
\item[(ii)] \(x \ast y \in A\) and \(y \in A\) imply that \(x \in A\).
\end{enumerate}

In what follows , let \(X\) denote a BCH-algebra unless otherwise specified.

A fuzzy set in \(X\) is a function \(\mu : X \to [0, 1]\).

A fuzzy set \(\mu\) in \(X\) is called a fuzzy subalgebra of \(X\) if

\[\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}, \forall x, y \in X,\]

and the complement of \(\mu\), denoted by \(\bar{\mu}\), is the fuzzy set in \(X\) given by \(\bar{\mu}(x) = 1 - \mu(x)\) for all \(x \in X\).

A mapping \(f : X \to Y\) of BCH-algebras is called a \textit{homomorphism} if

\[f(x \ast y) = f(x) \ast f(y)\]

\(\text{for all } x, y \in X.\)

\section*{Definition 2.1.} [1] By a \textit{t-norm} \(T\), we mean a function \(T : [0, 1] \times [0, 1] \to [0, 1]\) satisfying the following conditions:

\begin{align*}
\text{(T1)} & \quad T(x, 1) = x, \\
\text{(T2)} & \quad T(x, y) \leq T(x, z) \text{ if } y \leq z, \\
\text{(T3)} & \quad T(x, y) = T(y, x), \\
\text{(T4)} & \quad T(x, T(y, z)) = T(T(x, y), z),
\end{align*}

\(\text{for all } x, y, z \in [0, 1].\)

\section*{Proposition 2.2.} Every \textit{t-norm} \(T\) has a useful property:

\[T(\alpha, \beta) \leq \min(\alpha, \beta)\]

\(\text{for all } \alpha, \beta \in [0, 1].\)
Definition 2.3. [17] By a s-norm $S$, we mean a function $S : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following conditions:
\begin{enumerate}
  \item[(S1)] $S(x, 0) = x,$
  \item[(S2)] $S(x, y) \leq S(x, z)$ if $y \leq z,$
  \item[(S3)] $S(x, y) = S(y, x),$ 
  \item[(S4)] $S(x, S(y, z)) = S(S(x, y), z),$
\end{enumerate}
for all $x, y, z \in [0, 1]$.

Proposition 2.4. Every s-norm $S$ has a useful property:
\[ \max(\alpha, \beta) \leq S(\alpha, \beta) \]
for all $\alpha, \beta \in [0, 1]$.

For a t-norm (or s-norm) $P$ on $[0, 1]$, denote by $\Delta_P$ the set of element $\alpha \in [0, 1]$ such that $P(\alpha, \alpha) = \alpha$, i.e., $\Delta_P := \{ \alpha \in [0, 1] \mid P(\alpha, \alpha) = \alpha \}$.

Definition 2.5. Let $P$ be a t-norm (or s-norm). A fuzzy set $\mu$ in $X$ is said to satisfy idempotent property with respect to $P$ if $\text{Im}(\mu) \subseteq \Delta_P$.

Let $X$ denote a BCH-algebra. An intuitionistic fuzzy set (IFS for short) $A$ is an object having the form
\[ A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X \} \]
where the functions $\mu_A : X \rightarrow [0, 1]$ and $\gamma_A : X \rightarrow [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X \}$.

3 Intuitionistic $(T, S)$-normed fuzzy closed ideals

Definition 3.1. Let $T$ be a t-norm and $S$ be a s-norm on $[0, 1]$. An IFS $A = (\mu_A, \gamma_A)$ in $X$ is called an intuitionistic $(T, S)$-normed fuzzy subalgebra of BCH-algebra $X$ if
\begin{enumerate}
  \item[(F1)] $\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)),$
  \item[(F2)] $\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y)),$
\end{enumerate}
for all $x, y \in X$. 

Example 3.2. Let \( X = \{0, a, b, c, d\} \) be a BCH-algebra with the following Cayley table:

\[
\begin{array}{c|cccc}
\ast & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & a \\
b & b & b & 0 & 0 \\
c & c & c & c & 0 \\
d & d & d & d & d \\
\end{array}
\]

Let \( T : [0, 1] \times [0, 1] \rightarrow [0, 1] \) be a function defined by
\[
T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)
\]
for all \( \alpha, \beta \in [0, 1] \) and \( S : [0, 1] \times [0, 1] \rightarrow [0, 1] \) be a function defined by
\[
S(\alpha, \beta) = \min(\alpha + \beta, 1)
\]
for all \( \alpha, \beta \in [0, 1] \). Then \( T \) is a \( t \)-norm and \( S \) is a \( s \)-norm. Define an intuitionistic fuzzy set \( IFS \) \( A = (\mu_A, \gamma_A) \) by \( \mu_A(0) = \mu_A(d) = 0, \mu_A(a) = \mu_A(b) = \mu_A(c) = 0.09 \) and \( \gamma_A(a) = \gamma_A(b) = \gamma_A(c) = 0.9, \gamma_A(0) = \gamma_A(d) = 0.09 \). Then \( IFS \) \( A = (\mu_A, \gamma_A) \) is an intuitionistic \( (T, S) \)-normed fuzzy subalgebra of \( X \).

Theorem 3.3. IF \( \{A_i\} \) is a family of intuitionistic \( (T, S) \)-normed fuzzy subalgebra of \( X \), then \( \bigcap_{i \in I} A_i \) is an intuitionistic \( (T, S) \)-normed fuzzy subalgebra of \( X \), where \( \bigcap_{i \in I} A_i = (\bigvee \mu_i, \bigwedge \gamma_i) \).

Let \( \chi_A \) denote the characteristic function of a non-empty subset \( A \) of an BCH-algebra \( X \).

Theorem 3.4. If \( A \) is a subalgebra of an BCH-algebra \( X \), then the IFS \( \bar{I} = (\chi_A, \bar{\chi}_A) \) is an intuitionistic \( (T, S) \)-normed fuzzy subalgebra of \( X \).

Proof. Let \( x, y \in X \). If \( x, y \in A \), then \( x \ast y \in A \) since \( A \) is a subalgebra of \( X \). Hence
\[
\chi_A(x \ast y) = 1 \geq T(\chi_A(x), \chi_A(y)).
\]
Also, we have
\[
0 = 1 - \chi_A(x \ast y) = \bar{\chi}_A(x \ast y) \leq S(\bar{\chi}_A(x), \bar{\chi}_A(y)).
\]
If \( x \in A \) and \( y \notin A \), (or, \( x \notin A \) and \( y \in A \)), then \( \chi_A(x) = 1 \), or \( \chi_A(y) = 0 \). Thus we have
\[
\chi_A(x \ast y) \geq T(\chi_A(x), \chi_A(y)) = T(1, 0) = T(0, 1) = 0.
\]
Next we have
\[
S(\bar{\chi}_A(x), \bar{\chi}_A(y)) = S(1 - \chi_A(x), 1 - \chi_A(y)) = S(0, 1) = 1 \geq \bar{\chi}_A(x \ast y)
\]
. This proves the theorem.
**Theorem 3.5.** Let $A$ be a nonempty subset of a BCH-algebra $X$. If $\bar{A} = (\chi_A, \bar{\chi}_A)$ satisfies (F1) or (F2), then $\bar{A}$ is a subalgebra of a BCH-algebra $X$.

**Proof.** Suppose that $\bar{A} = (\chi_A, \bar{\chi}_A)$ satisfy (F1). Let $x, y \in A$. Then it follows from (F1) that

$$\chi_A(x * y) = T(\chi_A(x), \chi_A(y)) = T(1, 1) = 1$$

so that $\chi_A(x * y) = 1$, i.e., $x * y \in A$. Hence $A$ is a subalgebra of $X$. Now suppose that $\bar{A} = (\chi_A, \bar{\chi}_A)$ satisfy (F2). Let $x, y \in A$. Then from (F2), we have

$$\bar{\chi}_A(x * y) \leq S(\bar{\chi}_A(x), \bar{\chi}_A(y)) \leq S(1 - \chi_A(x), 1 - \chi_A(y)) = S(0, 0) = 0,$$

and thus $\bar{\chi}_A(x * y) = 1 - \chi_A(x * y) = 0$, i.e., $\chi_A(x * y) = 1$. This proves the theorem. \qed

**Definition 3.6.** Let $T$ be a $t$-norm and $S$ be a $s$-norm on $[0, 1]$. An intuitionistic $(T, S)$-normed fuzzy subalgebra $A = (\mu_A, \gamma_A)$ is called an intuitionistic idempotent $(T, S)$-normed fuzzy subalgebra of $X$ if $\mu_A$ and $\gamma_A$ satisfy the idempotent property with respect to $T$ and $S$, respectively.

**Example 3.7.** In Example 3.2, let $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a function defined by

$$T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$$

for all $\alpha, \beta \in [0, 1]$ and $S : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a function defined by

$$S(\alpha, \beta) = \min(\alpha + \beta, 1)$$

for all $\alpha, \beta \in [0, 1]$. Define an intuitionistic fuzzy set IFS $A = (\mu_A, \gamma_A)$ by $\mu_A(0) = \mu_A(d) = 1, \mu_A(a) = \mu_A(b) = \mu_A(c) = 0$ and $\gamma_A(a) = \gamma_A(b) = \gamma_A(c) = 1, \gamma_A(0) = \gamma_A(d) = 0$. Then IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic idempotent $(T, S)$-normed fuzzy subalgebra of $X$.

**Proposition 3.8.** Let $T$ be a $t$-norm and $S$ be a $s$-norm on $[0, 1]$. If IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic idempotent $(T, S)$-normed fuzzy subalgebra of BCH-algebra $X$, then $\mu_A(0 * x) \geq \mu_A(x)$ and $\gamma_A(0 * x) \leq \gamma_A(x)$ for all $x \in X$.

**Proof.** For any $x \in X$, we have

$$\mu_A(0 * x) \geq T((\mu_A(0), \mu_A(x))$$

$$\geq T(\mu_A(x), \mu_A(x)) \quad \text{[by (H1)]}$$

$$= T(T(\mu_A(x), \mu_A(x), \mu_A(x)) \quad \text{[by (T2)and (T3)]}$$

$$= \mu_A(x) \quad \text{[Since $\mu_A$ satisfies the idempotent property ]},$$
and
\[
\gamma_A(0 \ast x) \leq S((\gamma_A(0), \gamma_A(x))
\leq S(\gamma_A(x \ast x), \gamma_A(x)) \quad \text{[by (H1)]}
= S(S(\gamma_A(x), \gamma_A(x)), \gamma_A(x)) \quad \text{[by (S2) and (S3)]}
= \gamma_A(x) \quad \text{[Since \( \gamma_A \) satisfies the idempotent property ]},
\]
This completes the proof.

**Definition 3.9.** Let \( T \) be a \( t \)-norm and \( S \) be a \( s \)-norm on \([0, 1]\). An IFS \( A = (\mu_A, \gamma_A) \) in \( X \) is called an *intuitionistic \((T, S)\)-normed fuzzy closed ideal* of BCH-algebra \( X \) if
\[
\text{(F3) } \mu_A(0 \ast x) \geq \mu_A(x) \text{ and } \gamma_A(0 \ast x) \leq \gamma_A(x),
\text{(F4) } \mu_A(x) \geq T(\mu_A(x \ast y), \mu_A(y)) \text{ and } \gamma_A(x) \leq S(\gamma_A(x \ast y), \gamma_A(y)) \text{ for all } x, y \in X.
\]

Let \( T \) be a \( t \)-norm and \( S \) be a \( s \)-norm on \([0, 1]\). An intuitionistic \((T, S)\)-normed fuzzy closed ideal \( A = (\mu_A, \gamma_A) \) is called an *intuitionistic idempotent \((T, S)\)-normed fuzzy closed ideal* of \( X \) if \( \mu_A \) and \( \gamma_A \) satisfy the idempotent property with respect to \( T \) and \( S \), respectively.

**Example 3.10.** Let \( X = \{0, a, b, c\} \) be a BCH-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>c</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>0</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Define an intuitionistic fuzzy set \( A = (\mu_A, \gamma_A) \) by
\[
\mu_A(x) = \begin{cases} 
0.8 & \text{if } x \in \{0, c\}, \\
0.3 & \text{otherwise,}
\end{cases}
\quad \text{and} \quad
\gamma_A(x) = \begin{cases} 
0.3 & \text{if } x \in \{0, c\}, \\
0.8 & \text{otherwise.}
\end{cases}
\]

Let \( T : [0, 1] \times [0, 1] \rightarrow [0, 1] \) be a function defined by
\[
T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)
\]
and and \( S : [0, 1] \times [0, 1] \rightarrow [0, 1] \) be a function defined by
\[
S(\alpha, \beta) = \min(\alpha + \beta, 1)
\]
for all \( \alpha, \beta \in [0, 1] \). Then \( A = (\mu_A, \gamma_A) \) is an intuitionistic \((T, S)\)-normed fuzzy closed ideal of \( X \) which is not idempotent.
Example 3.11. In Example 3.10, define an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ by
\[
\mu_A(x) = \begin{cases} 
1 & \text{if } x \in \{0, c\}, \\
0 & \text{otherwise,}
\end{cases}
\quad \text{and} \quad \gamma_A(x) = \begin{cases} 
0 & \text{if } x \in \{0, c\}, \\
1 & \text{otherwise.}
\end{cases}
\]
Then $A = (\mu_A, \gamma_A)$ is an intuitionistic idempotent $(T, S)$-normed fuzzy closed ideal of $X$.

Theorem 3.12. Every intuitionistic idempotent $(T, S)$-normed fuzzy subalgebra satisfying (F4) is an intuitionistic idempotent $(T, S)$-normed fuzzy closed ideal.

Proof. Using Proposition 3.8, it is straightforward. \qed

Proposition 3.13. If IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic idempotent $(T, S)$-normed fuzzy closed ideal of BCH-algebra $X$, then $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(x) \leq \gamma_A(0)$ for all $x \in X$.

Proof. Using (F3), (F4), (T2) and (S2), we have
\[
\mu_A(0) \geq T(\mu_A(0 \ast x), \mu_A(x)) \geq T(\mu_A(x), \mu_A(x)) = \mu_A(x)
\]
and
\[
\gamma_A(0) \leq S(\gamma_A(0 \ast x), \gamma_A(x)) \leq S(\gamma_A(x), \gamma_A(x)) = \gamma_A(x)
\]
for all $x \in X$, completing the proof. \qed

Theorem 3.14. Every intuitionistic $(T, S)$-normed fuzzy closed ideal is an intuitionistic $(T, S)$-normed fuzzy subalgebra.

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic $(T, S)$-normed fuzzy closed ideal of $X$ and let $x, y \in X$. Then
\[
\mu_A(x \ast y) \geq T(\mu_A(x \ast y \ast x), \mu_A(x)) \quad \text{[by (F4)]}
\geq T(\mu_A(x \ast x \ast y), \mu_A(x)) \quad \text{[by (H3)]}
= T(\mu_A(0 \ast y), \mu_A(x)) \quad \text{[by (H1)]}
\geq T(\mu_A(x), \mu_A(y)) \quad \text{[by (F3), (T2) and (T3)]}
\]
and
\[
\gamma_A(x \ast y) \leq S(\gamma_A(x \ast y \ast x), \gamma_A(x)) \quad \text{[by (F4)]}
\leq S(\gamma_A(x \ast x \ast y), \gamma_A(x)) \quad \text{[by (H3)]}
= S(\gamma_A(0 \ast y), \gamma_A(x)) \quad \text{[by (H1)]}
\leq S(\gamma_A(x), \gamma_A(y)) \quad \text{[by (F3), (S2) and (S3)]}
\]
Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic $(T, S)$-normed fuzzy subalgebra of $X$ \qed
The converse of Theorem 3.14 may not be true. For example, the intuitionistic $(T, S)$-normed fuzzy subalgebra in Example 3.2 is not a intuitionistic $(T, S)$-normed fuzzy closed ideal since

$$
\mu_A(a) = 0.09 < 0.9 = T(\mu_A(a * d), \mu_A(d)).
$$

We give a condition for an intuitionistic $(T, S)$-normed fuzzy subalgebra to be an intuitionistic $(T, S)$-normed fuzzy closed ideal.

**Theorem 3.15.** Let $A = (\mu_A, \gamma_A)$ be an intuitionistic $(T, S)$-normed fuzzy subalgebra of $X$. If $A = (\mu_A, \gamma_A)$ satisfies the idempotent property and inequalities $\mu_A(x * y) \leq \mu_A(y * x)$ and $\gamma_A(x * y) \geq \gamma_A(y * x)$ for all $x, y \in X$, then $A = (\mu_A, \gamma_A)$ is an intuitionistic $(T, S)$-normed fuzzy closed ideal of $X$.

**Proof.** Let $A = (\mu_A, \gamma_A)$ be an intuitionistic $(T, S)$-normed fuzzy subalgebra of $X$ which satisfies the inequalities

$$
\mu_A(x * y) \leq \mu_A(y * x) \text{ and } \gamma_A(x * y) \geq \gamma_A(y * x)
$$

for all $x, y \in X$. It follows from Proposition 3.8 that $\mu_A(0 * x) \geq \mu_A(x)$ and $\gamma_A(0 * x) \leq \gamma_A(x)$ for all $x, y \in X$. Then

$$
\mu_A(x) = \mu_A(x * 0) \geq \mu_A(0 * x) = \mu_A((y * y) * x)
$$

$$
= \mu_A((y * x) * y) \geq T(\mu_A(y * x), \mu_A(y)) = T(\mu_A(x * y), \mu_A(y)),
$$

and

$$
\gamma_A(x) = \gamma_A(x * 0) \leq \gamma_A(0 * x) = \gamma_A((y * y) * x)
$$

$$
= \gamma_A((y * x) * y) \leq S(\gamma_A(y * x), \gamma_A(y)) \leq S(\gamma_A(x * y), \gamma_A(y)),
$$

Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic $(T, S)$-normed fuzzy closed ideal of $X$. \qed

Let $A = (\mu_A, \gamma_A)$ be an IFS in $X$ and let $\alpha \in [0, 1]$. Then the sets

$$
U(\mu_A; \alpha) := \{x \in X : \mu_A(x) \geq \alpha\}
$$

and

$$
L(\gamma_A; \alpha) := \{x \in X : \gamma_A(x) \leq \alpha\}
$$

are called a $\mu$-level $\alpha$-cut and a $\gamma$-level $\alpha$-cut of $A$, respectively.

**Theorem 3.16.** Let $T$ be a $t$-norm and $S$ be a $s$-norm let $A = (\mu_A, \gamma_A)$ be an IFS in $X$ such that the non-empty sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \alpha)$ are closed ideals of $X$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic $(T, S)$-normed fuzzy closed ideal of $X$. 

Proof. Suppose that there exists $x_0, y_0 \in X$ such that

$$\mu_A(x_0) < T(\mu_A(x_0 \ast y_0), \mu_A(y_0)).$$

Taking $\alpha_0 := \frac{1}{2}(\mu_A(x_0) + T(\mu_A(x_0 \ast y_0), \mu_A(y_0)))$, then

$$\min(\mu_A(x_0 \ast y_0), \mu_A(y_0)) \geq T(\mu_A(x_0 \ast y_0), \mu_A(y_0)) \geq \alpha_0 > \mu_A(x_0).$$

It follows that $x_0 \ast y_0, y_0 \in U(\mu_A; \alpha_0)$ and $x_0 \notin U(\mu_A; \alpha_0)$. This is a contradiction and hence $\mu_A$ satisfies the inequality $\mu_A(x) \geq T(\mu_A(x \ast y), \mu_A(y))$ for all $x, y \in X$. Similarly, suppose that there exists $x_0, y_0 \in X$ such that

$$\gamma_A(x_0) > S(\gamma_A(x_0 \ast y_0), \gamma_A(y_0)).$$

Taking $\beta_0 := \frac{1}{2}(\gamma_A(x_0) + S(\gamma_A(x_0 \ast y_0), \gamma_A(y_0)))$, then

$$\max(\gamma_A(x_0 \ast y_0), \gamma_A(y_0)) \leq S(\gamma_A(x_0 \ast y_0), \gamma_A(y_0)) \leq \beta_0 < \gamma_A(x_0).$$

It follows that $x_0 \ast y_0, y_0 \in L(\gamma_A; \beta_0)$ and $x_0 \notin L(\gamma_A; \beta_0)$. This is a contradiction and hence $\gamma_A$ satisfies the inequality $\gamma_A(x) \leq S(\gamma_A(x \ast y), \gamma_A(y))$ for all $x, y \in X$. Now assume that there exists $x_0 \in X$ such that $\mu_A(0 \ast x_0) < \mu_A(x_0)$. Taking

$$\alpha_0 := \frac{1}{2}(\mu_A(0 \ast x_0) + \mu_A(x_0))$$

then $\mu_A(0 \ast x_0) \leq \alpha_0$ and $\mu_A(x_0) \geq \alpha_0$. It follows that $x_0 \in U(\mu_A; \alpha_0)$ but $0 \ast x_0 \notin U(\mu_A; \alpha_0)$. This is a contraction. Hence $\mu_A(0 \ast x) \geq \mu_A(x)$ for all $x \in X$. Similarly, we get $\gamma_A(0 \ast x) \leq \gamma_A(x)$ for all $x \in X$. \qed

References


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