Mixed Multivalued Variational-Like Inequalities

Shamshad Husain

Department of Engineering, Salalah College of Technology
Post Box: 608, Salalah-211, Sultanate of Oman
s_husain123@yahoo.co.in

Abstract

In this paper, we consider a new concept of relaxed $\eta$-$\alpha$-pseudomonotonicity with respect to $g$. By using KKM technique, we proved some existence results for mixed multivalued variational inequalities in reflexive Banach spaces.

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1. Introduction

Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. This theory has been generalized and extended in many directions using novel and innovative techniques. In recent years, considerable interest has been shown in developing various classes of variational inequalities both for its own sake and for its applications. Several numerical methods have been developed for solving variational inequalities and related optimization problems. See [1,3,6,7,8,9] and references therein.

Inspired and motivated by [4,6,8] and references therein, in this paper, we consider mixed multivalued variational-like inequalities in the Banach spaces.

2. Preliminaries

Let $E$ be a real reflexive Banach space with dual space $E^*$, and $\langle \cdot, \cdot \rangle$ denote the pairing between $E$ and $E^*$. Let $K$ be a nonempty subset of $E$, and $2^E$ denote the family of all the nonempty subset of $E$. 

Definition 2.1. Let \( g : K \rightarrow K, \eta : K \times K \rightarrow E \) be the mappings. A mapping \( T : K \rightarrow 2^{E^*} \) is said to be relaxed \( \eta\alpha \)-pseudomonotone with respect to \( g \) if there exists a function \( \alpha : E \rightarrow R \) with \( \alpha(tz) = t^p\alpha(z) \) for all \( t > 0 \) and \( z \in E \) such that for any \( x, y \in K \), we have
\[
\langle u, \eta(g(x), y) \rangle \geq 0 \Rightarrow \langle v, \eta(g(x), y) \rangle \geq \alpha(g(x) - y), \quad \forall \ u \in T(x), \ v \in T(y)
\]
where \( p > 1 \) is a constant.

If \( g \equiv I \), \( T \) is single-valued mapping, then our Definition 2.1 coincides with Definition 2.1 of [4].

The followin definitions are important for the proof of main results of this paper.

Definition 2.2. Let \( g : K \rightarrow K \) and \( \eta : K \times K \rightarrow E \) be two mappings. A map \( T : K \rightarrow 2^{E^*} \) is said to be \( \eta \)-hemicontinuous with respect to \( g \), if for any fixed \( x, y \in K, t > 0, u_\alpha \in T(g(x) + \alpha(y - g(x)), \eta(g(x), y)) \) there exists \( u_0 \in T(x) \) such that
\[
\langle u_\alpha, \eta(g(x), y) \rangle \rightarrow \langle u_0, \eta(g(x), y) \rangle \quad \text{as} \quad t \rightarrow 0^+.
\]

Definition 2.3. A mapping \( F : K \rightarrow 2^E \) is said to be a KKM mapping if for any \( \{x_1, \ldots, x_n\} \subset K, \text{Co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n F(x_i) \), where \( \text{Co}\{x_1, \ldots, x_n\} \) denotes the convex hull of \( x_1, \ldots, x_n \).

Lemma 2.1. Let \( K \) be a nonempty subset of a Hausdorff topological vector space \( E \) and let \( F : K \rightarrow 2^E \) be a KKM mapping. If \( F(x) \) is closed in \( E \) for every \( x \in K \) and compact for some \( x_0 \in K \), then \( \bigcap_{x \in K} F(x) \neq \emptyset \).

3. Existence Results

In this section, we consider the following mixed multivalued variational-like inequality problem:

Let \( g : K \rightarrow K, \eta : K \times K \rightarrow E \) and \( T : K \rightarrow 2^{E^*} \) be the mapping. Find \( x \in K, u \in T(x) \) such that
\[
\langle u, \eta(y, g(x)) \rangle \geq 0, \quad \text{for all} \quad y \in K
\]
where \( K \) is a nonempty closed convex subset of a real reflexive Banach space \( E \).

Theorem 3.1. Let \( K \) be a nonempty closed convex subset of a real reflexive Banach space \( E \) and \( g : K \rightarrow K \) be the mapping. Let \( T : K \rightarrow 2^{E^*} \) be a \( \eta \)-hemicontinuous and relaxed \( \eta\alpha \)-pseudomonotone with respect to \( g \). Assume that
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(i) \( \eta(g(x), x) = 0 \) for all \( x \in K \);

(ii) for any fixed \( y \in K \) and \( u \in T(x) \), the set \( \langle u, \eta(g(x), y) \rangle \) is convex.

Then the set of elements \( (x, u) \) is a solution of (3.1) if and only if

\[
\langle v, \eta(y, g(x)) \rangle \geq \alpha(y - g(x)), \quad \text{for all } y \in K.
\]

(3.2)

Proof. Suppose that \( (x, u) \) is a solution of (3.1). Since \( T \) is relaxed \( \eta \)-\( \alpha \)-pseudomonotone with respect to \( g \), we have

\[
\langle v, \eta(y, g(x)) \rangle \geq \alpha(y - g(x)), \quad \text{for all } y \in K.
\]

Hence \( (x, u) \) is a solution of (3.2).

Conversely, suppose that \( (x, u) \) is a solution of (3.2) and \( y \in K \) be any point. Letting \( x_t = ty + (1 - t)g(x) \), \( t \in (0, 1] \) such that \( u_t \in T(x_t) \) then \( x_t \in K \).

It follows from (3.2) that

\[
\langle u_t, \eta(x_t, g(x)) \rangle \geq \alpha(x_t - g(x)) \]
\[
= \alpha(t(y - g(x))) \]
\[
= t^p \alpha(y - g(x)) \quad \text{(3.3)}
\]

By condition (i) and (ii), we have

\[
\langle u_t, \eta(x_t, g(x)) \rangle \leq \langle u_t, \eta(y, g(x)) \rangle + (1 - t)\langle u_t, \eta(x, g(x)) \rangle \]
\[
\leq t\langle u_t, \eta(y, g(x)) \rangle \quad \text{(3.4)}
\]

It follows from (3.3) and (3.4) that

\[
\langle u_t, \eta(x_t, g(x)) \rangle \geq t^{p-1} \alpha(y - g(x)) \quad \text{(3.5)}
\]

for all \( y \in K \). Since \( T \) is \( \eta \)-hemicontinuous with respect to \( g \) and \( p > 1 \), letting \( t \to 0^+ \) in (3.5), we get

\[
\langle u, \eta(y, g(x)) \rangle \geq 0, \quad \text{for all } y \in K.
\]

Therefore \( (x, u) \) is a solution of (3.1).

Theorem 3.2. Let \( K \) be a bounded closed convex subset of the reflexive Banach space \( E \). Let \( g : K \to K \), \( \eta : K \times K \to E \) be the mapping and \( T : K \to 2^{E^*} \) be a \( \eta \)-hemicontinuous and relaxed \( \eta \)-\( \alpha \)-pseudomonotone with respect to \( g \). Assume that
(i) $\eta(g(x), y) + \eta(y, g(x)) = 0$, for all $x, y \in K$;

(ii) for any fixed $y \in K$ and $u \in T(x)$, the set $\langle u, \eta(g(x), y) \rangle$ is convex and lower semicontinuous;

(iii) $\alpha : E \to R$ is weakly lower semicontinuous, i.e. for any net $\{x_\beta\} \subset E$, $x_\beta$ converges weakly to $x_0$ implies that $\alpha(x_0) \leq \liminf_{\beta} \alpha(x_\beta)$.

Then there exists atleast one solution for (3.1).

**Proof.** For any $y \in K$ define two set-valued mappings $P, Q : K \to 2^K$ as follows:

$$P(y) = \{x \in K : \exists u \in T(x) \text{ such that } \langle u, \eta(y, g(x)) \rangle \geq 0\}$$

$$Q(y) = \{x \in K : \exists v \in T(y) \text{ such that } \langle v, \eta(y, g(x)) \rangle \geq \alpha(y - g(x))\}.$$ 

We claim that $P$ is a KKM mapping. If $P$ is not a KKM mapping, then there exists $\{y_1, \ldots, y_n\} \subset K$ such that $\text{co}\{y_1, \ldots, y_n\} \not\subset \bigcup_{i=1}^n P(x_i)$, i.e., there exists a $y_0 \in \text{co}\{y_1, \ldots, y_n\}$, $y_0 = \sum_{i=1}^n t_i x_i$, where $t_i \geq 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n t_i = 1$, but $y_0 \not\in \bigcup_{i=1}^n P(x_i)$. By the definition of $P$, we have

$$0 = \langle u_0, \eta(y_0, g(y_0)) \rangle = \langle u_0, \eta(\sum_{i=1}^n t_i y_i, g(y_0)) \rangle$$

$$\leq \sum_{i=1}^n t_i \langle u_0, \eta(y_i, g(y_0)) \rangle$$

$$< 0$$

which is a contradiction. This implies that $P$ is a KKM mapping. Now we prove that $P(y) \subset Q(y)$, for all $y \in K$.

For any given $y \in K$, letting $x \in P(y)$, then

$$\langle u, \eta(y, g(x)) \rangle \geq 0.$$ 

Since $T$ is relaxed $\eta$-$\alpha$-pseudomonotone with respect to $g$, we have there exists $v \in T(y)$ such that

$$\langle v, \eta(y, g(x)) \rangle \geq \alpha(y - g(x)).$$

It follows that $x \in Q(y)$ and so

$$P(y) \subset Q(y), \text{ for all } y \in K.$$
This implies that $Q$ is also a KKM mapping.

Since the set $\langle u, \eta(g(x), y) \rangle$ is convex, lower semicontinuous, we know that it is weakly the lower semicontinuous. From the definition of $Q$ and the weakly lower semicontinuity of $\alpha$, it is easy to see that $P(y)$ is weakly closed for all $y \in K$. Since $K$ is bounded closed and convex, we know that $K$ is weakly compact, and so $Q(y)$ is weakly compact in $K$ for each $y \in K$. Therefore, the conditions of Lemma 2.1 are satisfied in the weak topology. It follows from Lemma 2.1 and Theorem 3.1 that

$$\bigcap_{y \in K} P(y) = \bigcap_{y \in K} Q(y) \neq \emptyset.$$  

Hence, there exists $x \in K$ and $u \in T(x)$ such that

$$\langle u, \eta(y, g(x)) \rangle \geq 0, \quad \text{for all } y \in K.$$  

This completes the proof.

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References


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