On KS-Semigroup Homomorphism

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Abstract

A BCK-algebra is an algebraic structure of a set $X$ with one binary operation. A KS-semigroup is a semigroup with respect to one binary operation and also a BCK-algebra with respect to another binary operation satisfying left and right distributive laws.

This paper characterizes KS-semigroup homomorphisms and proves the isomorphism theorems for KS-semigroups.

Keywords: BCK algebra, semigroups, KS-semigroups, KS-semigroup homomorphism, ideals

1 Introduction

In abstract algebra, mathematical systems with one binary operation called group and two binary operations called rings were investigated. In 1966, Y. Imai and K. Iseki defined a class of algebra called BCK-algebra in their paper entitled “On Axioms of Propositional Calculi XI”. A BCK-algebra is named after the combinators B, C and K by Carew Arthur Merideth, an Irish logician. At the same time, Iseki introduced another class of algebra called BCI-algebra, which is a generalization of the class of BCK-algebra and investigated its properties in his paper entitled “An Algebra Related to Propositional Calculus XII”. In characterizing BCK-algebra/BCI-algebra, the ideals play an important role. In 2006, Kyung Ho Kim[5] introduced a new class of algebraic structure called KS-semigroup in his paper entitled “On Structure of KS-semigroup”. He characterized ideals of a KS-semigroup and proved the first isomorphism theorem for KS-semigroups.

2 Basic Definitions

Definition 1 [5] A $BCI$-algebra is a triple $(X, *, 0)$, where $X$ is a nonempty set, $*$ is a binary operation on $X$, $0 \in X$ is an element such that the following axioms are satisfied for every $x, y, z \in X$: 
i. \([(x * y) * (x * z)] * (z * y) = 0;
ii. \[x * (x * y)] * y = 0;
iii. \[x * x = 0;
iv. \[x * y = 0 \text{ and } y * x = 0 \text{ implies } x = y.

If \(0 * x = 0\) for all \(x \in X\), then \(X\) is called a \(BCK\)-algebra.

**Example 2** Let \((X, +)\) be an abelian group. Define the binary operation \(x * y = x - y\) for all \(x, y \in X\). Then \((X, *, 0)\) is a BCI-algebra which is not a BCK-algebra since \(0 * x = -x \neq 0\) for all \(x \in X\).

A BCK-algebra satisfies the following properties: \(x * 0 = x\) and \((x * y) * z = (x * z) * y\) for all \(x, y, z \in X\).

**Definition 3** [2] A *semigroup* is an ordered pair \((S, \cdot)\), where \(S\) is a nonempty set and “\(\cdot\)” is an associative binary operation on \(S\).

**Definition 4** [5] A *KS-semigroup* is a nonempty set \(X\) together with two binary operations “\(*\)” and “\(\cdot\)” and constant 0 satisfying the following:

i. \((X, *, 0)\) is a BCK-algebra;
ii. \((X, \cdot)\) is a semigroup; and
iii. The operation “\(\cdot\)” is left and right distributive over the operation “\(*\)”, that is, \(x \cdot (y * z) = (x \cdot y) * (x \cdot z)\) and \((x * y) \cdot z = x \cdot z * y \cdot z\) for all \(x, y, z \in X\).

**Example 5** Let \(X = \{0, a, b, c\}\). Define the operation “\(*\)” by the following tables.

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
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<td>c</td>
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<th>0</th>
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Then by routine calculations, \(X\) can be shown to be a KS-semigroup.

In what follows, we shall write the multiplication \(x \cdot y\) by \(xy\) for convenience.

**Definition 6** [5] A nonempty subset \(A\) of a semigroup \((X, \cdot)\) is said to be *left* (resp. *right*) stable if \(xa \in A\) (resp. \(ax \in A\)) whenever \(x \in X\) and \(a \in A\). Both left and right stable is a *two sided stable* or *simply stable*.

**Definition 7** [5] A nonempty subset \(A\) of a KS-semigroup \(X\) is called a *left* (resp. *right*) *ideal* of \(X\) if
i. $A$ is left (resp. right) stable subset of $(X, \cdot)$;
ii. for any $x, y \in X$, $x \ast y \in A$ and $y \in A$ imply that $x \in A$.

A subset which is both left and right ideal is called a *two sided ideal* or simply an *ideal*.

In a KS-semigroup $X$, $x0 = 0x = 0$ for all $x \in X$. Thus, if $A$ is an ideal of $X$, $0 = 0a \in A$ for any $a \in A$.

**Definition 8** [6] A nonempty subset $S$ of a KS-semigroup $X$ with binary operations “$\ast$” and “$\cdot$” is called a *sub KS-semigroup* if $x \ast y \in S$ and $xy \in S$ for all $x, y \in S$.

Observe that an ideal of a KS-semigroup is a sub KS-semigroup, hence, it is also a KS-semigroup.

**Definition 9** [5] A nonempty subset $A$ of a KS-semigroup $X$ is called a *left* (resp. right) *$P$-ideal* of $X$ if

(i) $A$ is a left (resp. right) stable subset of $(X, \cdot)$;
(ii) For any $x, y, z \in X$, $(x \ast y) \ast z \in A$ and $y \ast z \in A$ imply that $x \ast z \in A$.

A subset of $X$ which is both left and right $P$-ideal is called a *$P$-ideal* of a KS-semigroup $X$. A $P$-ideal is always an ideal, whence a sub KS-semigroup.

**Definition 10** [5] Let $X$ and $X'$ be KS-semigroups and $f : X \rightarrow X'$. Then

i. $f$ is a *homomorphism* if $f(x \ast y) = f(x) \ast f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in X$.
ii. $f$ is a *monomorphism* if $f$ is a one-to-one homomorphism.
iii. $f$ is an *epimorphism* if $f$ is an onto homomorphism.
iv. $f$ is an *isomorphism* if $f$ is a bijective homomorphism.

**Definition 11** Let $f : X \rightarrow Y$ be a KS-semigroup homomorphism. The *kernel* of $f$ is the set $Ker f = \{x \in X : f(x) = 0\}$ and the *image* of $f$ is the set $Im f = \{f(x) \in Y : x \in X\}$.

**Definition 12** The relation $x \leq y$ if and only if $x \ast y = 0$ is a partial order and is called the *natural order* on a BCK-algebra $X$.

Note that $Ker f$ is an ideal of the domain of $f$ and that the following lemma holds in [5].

**Lemma 13** Let $f : X \rightarrow X'$ be a KS-semigroup homomorphism. Then $f(0) = 0$ and $x \leq y$ implies $f(x) \leq f(y)$.
**Theorem 14** Let $f : X \rightarrow X'$ be a KS-semigroup homomorphism. Then $f$ is a monomorphism if and only if $\ker f = \{0\}$.

*Proof.* Let $f : X \rightarrow X'$ be a KS-semigroup homomorphism and suppose that $f$ is a monomorphism. Let $x \in \ker f$. Then $f(x) = 0 = f(0)$ by Lemma 13. Since $f$ is one-to-one, $x = 0$. Therefore, $\ker f = \{0\}$. Conversely, suppose that $\ker f = \{0\}$ and $f(x) = f(y)$ for any $x, y \in X$. By Definition 1(iv), $f(x) * f(y) = 0$ and $f(y) * f(x) = 0$. Since $f$ is a homomorphism, $f(x * y) = 0$ and $f(y * x) = 0$. Thus, $x * y, y * x \in \ker f = \{0\}$. Hence, $x * y = 0$ and $y * x = 0$. Therefore, $x = y$ and $f$ is one-to-one. \qed

**Theorem 15** Let $X, Y, Z$ be KS-semigroups. Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are KS-semigroup homomorphisms. Then $g \circ f : X \rightarrow Z$ is also a KS-semigroup homomorphism.

*Proof.* Assume that $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are KS-semigroup homomorphisms. Define $g \circ f : X \rightarrow Z$ by $(g \circ f)(x) = g(f(x))$. Now, $(g \circ f)(x * y) = g(f(x * y)) = g(f(x) * f(y)) = g(f(x)) * g(f(y)) = (g \circ f)(x) * (g \circ f)(y)$ and $(g \circ f)(xy) = g(f(xy)) = g(f(x)f(y)) = g(f(x))g(f(y)) = (g \circ f)(x)(g \circ f)(y)$. Therefore, $g \circ f$ is a homomorphism. \qed

**Theorem 16** Let $f : X \rightarrow X'$ be a KS-semigroup homomorphism. If $B$ is an ideal of $X'$, then $f^{-1}(B) = \{a \in X \mid f(a) \in B\}$ is an ideal of $X$ containing $\ker f$.

*Proof.* Let $f : X \rightarrow X'$ be a KS-semigroup homomorphism. Suppose that $B$ is an ideal of $X'$. Let $x \in X$ and $z \in f^{-1}(B)$. Then $f(z) \in B$ and $f(xz) = f(x)f(z) \in B$ and $f(zx) = f(z)f(x) \in B$ since $B$ is stable. Thus, $xz, zx \in f^{-1}(B)$ and $f^{-1}(B)$ is stable. Let $x, y \in X$ such that $y \in f^{-1}(B)$ and $x * y \in f^{-1}(B)$. Then $f(y) \in B$ and $f(x) * f(y) = f(x * y) \in B$. Since $B$ is an ideal, $f(x) \in B$, that is, $x \in f^{-1}(B)$. Therefore, $f^{-1}(B)$ is an ideal of $X$. Moreover, $\{0\} \subseteq B$ implies that $\ker f = f^{-1}(\{0\}) \subseteq f^{-1}(B)$. \qed

**Theorem 17** Let $f : X \rightarrow X'$ be a KS-semigroup epimorphism.

i. If $A$ is an ideal of $X$, then $f(A)$ is an ideal of $X'$.

ii. If $A$ is a $P$-ideal of $X$, then $f(A)$ is a $P$-ideal of $X'$.

*Proof.* Let $f : X \rightarrow X'$ be a KS-semigroup epimorphism.
i. Let \( f(a) \in f(A) \) and \( y \in X' \). Since \( f \) is onto, there exists \( x \in X \) such that \( f(x) = y \). Thus, \( ax, xa \in A \) implies that \( f(ax), f(xa) \in f(A) \) and so \( f(A) \) is a stable subset of \( X' \). Now, suppose that \( f(y) \in f(A) \) and \( f(x) * f(y) \in f(A) \). Then \( y \in A \) and \( f(x * y) = f(x) * f(y) \in f(A) \) imply \( x * y \in A \). Since \( A \) is an ideal of \( X \), \( y \in A \), \( x * y \in A \) imply \( x \in A \). Therefore, \( f(x) \in f(A) \) and \( f(A) \) is an ideal of \( X' \).

ii. Suppose that \( A \) is a \( P \)-ideal of \( X \). Then \( A \) is an ideal of \( X \) and by (i), \( f(A) \) is an ideal of \( X' \). Thus, \( f(A) \) is a stable subset of \( X' \). Let \( f(x), f(y), f(z) \in f(A) \) for some \( x, y, z \in A \) such that \( [f(x) * f(y)] * f(z) \in f(A) \) and \( f(y) * f(z) \in f(A) \). Since \( f \) is a homomorphism, \( f([x*y]*z) \in f(A) \) and \( f(y*z) \in f(A) \). Thus, \( (x*y)*z \in A \) and \( y*z \in A \). Since \( A \) is a \( P \)-ideal, \( x * z \in A \). Thus, \( f(x) * f(z) = f(x * z) \in f(A) \). Therefore, \( f(A) \) is a \( P \)-ideal of \( X' \).

**Definition 18** [1] A BCK-algebra is said to be

i. commutative if \( x * (x * y) = y * (y * x) \) for any \( x, y \in X \).

ii. implicative if \( x * (y * x) = x \) for all \( x, y, z \in X \).

iii. positive implicative if \( (x * y) * z = (x * z) * (y * z) \), for all \( x, y, z \in X \).

**Theorem 19** Let \( f : X \rightarrow X' \) be a KS-semigroup homomorphism such that \( X \) is positive implicative. Then \( \text{Ker} f \) is a \( P \)-ideal of \( X \).

**Proof.** Let \( f : X \rightarrow X' \) be a KS-semigroup homomorphism such that \( X \) is positive implicative. Since \( \text{Ker} f \) is an ideal of \( X \), it follows that \( \text{Ker} f \) is a stable subset of \( X \). Let \( x, y, z \in X \) such that \( (x * y) * z \in \text{Ker} f \) and \( y * z \in \text{Ker} f \). By a property of a BCK algebra, \( (x * z) * y = (x * y) * z \in \text{Ker} f \). Hence, \( 0 = [f(x) * f(z)] * f(y) \) and \( 0 = f(y*z) = f(y) * f(z) \). By Definition 12, this implies that \( f(x) * f(z) \leq f(y) \) and \( f(y) \leq f(z) \). By transitivity, \( f(x) * f(z) \leq f(z) \) and so \( [f(x) * f(z)] * f(z) = 0 \). Since \( f \) is a homomorphism and \( X \) is positive implicative, \( 0 = f((x * z) * z) = f((x * z) * (z * z)) = f((x * z) * 0) = f(x * z) \). Thus, \( x * z \in \text{Ker} f \). Therefore, \( \text{Ker} f \) is a \( P \)-ideal of \( X \). \( \square \)

**Definition 20** A strong KS-semigroup is a KS-semigroup \( X \) satisfying \( x * y = x * xy \) for each \( x, y \in X \).

**Definition 21** The element 1 is called a unity in a KS-semigroup \( X \) if \( 1x = x1 = x \) for all \( x \in X \).

**Theorem 22** Let \( f : X \rightarrow X' \) be a KS-semigroup homomorphism.

i. If \( X \) is commutative, then \( f(X) \) is commutative.
ii. If $X$ is implicative, then $f(X)$ is implicative.

iii. If $X$ is positive implicative, then $f(X)$ is positive implicative.

iv. If $X$ is a strong KS-semigroup with unity, then $f(X)$ is a strong KS-semigroup with unity.

**Proof.** Let $f : X \rightarrow X'$ be a KS-semigroup homomorphism. Let $f(x), f(y), f(z) \in f(X)$ for some $x, y, z \in X$.

(i) Suppose that $X$ is commutative. Then $f(x) \ast [f(x) \ast f(y)] = f(x) \ast f(x \ast y) = f(x \ast (x \ast y)) = f(y \ast (y \ast x)) = f(y) \ast f(y \ast x) = f(y) \ast [f(y) \ast f(x)]$. Therefore, $f(X)$ is also a commutative KS-semigroup.

(ii) Suppose that $X$ is implicative. Then $f(x) \ast [f(y) \ast f(x)] = f(x) \ast f(y \ast x) = f[x \ast (y \ast x)] = f(x)$. Therefore, $f(X)$ is implicative.

(iii) Suppose that $X$ is positive implicative. Then $[f(x) \ast f(y)] \ast f(z) = f(x \ast y) \ast f(z) = f((x \ast y) \ast z) = f((x \ast z) \ast (y \ast z)) = f(x \ast z) \ast f(y \ast z) = [f(x) \ast f(z)] \ast [f(y) \ast f(z)]$. Therefore, $f(X)$ is positive implicative.

(iv) Suppose that $X$ is a strong KS-semigroup with unity. Then $f(x) \ast f(y) = f(x \ast y) = f(x \ast xy) = f(x) \ast f(xy) = f(x) \ast f(x)f(y)$ and $f(1)f(x) = f(1x) = f(x) = f(x1) = f(x)f(1)$. Therefore, $f(X)$ is a strong KS-semigroup with unity $f(1)$. □

**Theorem 23** Let $f : X \rightarrow X'$ be a KS-semigroup monomorphism.

i. If $f(X)$ is commutative, then $X$ is commutative.

ii. If $f(X)$ is implicative, then $X$ is implicative.

iii. If $f(X)$ is positive implicative, then $X$ is positive implicative.

iv. If $f(X)$ is a strong KS-semigroup with unity, then $X$ is a strong KS-semigroup with unity.

**Proof.** Let $f : X \rightarrow X'$ be a KS-semigroup monomorphism and let $x, y, z \in Z$. Then $f(x), f(y), f(z) \in f(X)$.

i. Suppose that $f(X)$ is commutative. Then $f(x) \ast [f(x) \ast f(y)] = f(y) \ast [f(y) \ast f(x)]$. Since $f$ is a homomorphism, $f[x \ast (x \ast y)] = f[y \ast (y \ast x)]$. Also, since $f$ is one-to-one, $x \ast (x \ast y) = y \ast (y \ast x)$. Therefore, $X$ is commutative.

ii. Suppose that $f(X)$ is implicative. Then $f(x) \ast [f(y) \ast f(x)] = f(x)$. Since $f$ is a homomorphism, $f[x \ast (y \ast x)] = f(x)$. Also, since $f$ is one-to-one, $x \ast (y \ast x) = x$. Thus, $X$ is an implicative KS-semigroup.

iii. Suppose that $f(X)$ is positive implicative. Then $[f(x) \ast f(y)] \ast f(z) = [f(x) \ast f(z)] \ast [f(y) \ast f(z)]$. Since $f$ is a homomorphism, this implies that $f[(x \ast y) \ast z] = f[(x \ast z) \ast (y \ast z)]$. Also, since $f$ is one-to-one, $(x \ast y) \ast z = (x \ast z) \ast (y \ast z)$. Therefore, $X$ is positive implicative.
iv. Suppose that \( f(X) \) is a strong KS-semigroup with unity. Then \( f(x) \ast f(y) = f(x) \ast f(x)f(y) \) and \( f(1)f(x) = f(x)f(1) = f(x) \). Since \( f \) is a homomorphism, \( f(x \ast y) = f(x \ast xy) \) and \( f(1x) = f(x1) = f(x) \). Also, since \( f \) is one-to-one, \( x \ast y = x \ast xy \) and \( 1x = x1 = x \). Therefore, \( X \) is a strong KS-semigroup with unity.

Let \( M \) and \( N \) be ideals of a KS-semigroup \( X \). Define \( MN = \{mn \in MN : m \in M, n \in N\} \).

**Lemma 24** Let \( M \) and \( N \) be ideals of a KS-semigroup \( X \) with unity such that \( 1 \in M \cap N \). Then \( MN \) is an ideal of \( X \).

**Proof.** Assume that \( M \) and \( N \) are ideals of a KS-semigroup \( X \). Let \( x \in X \) and \( mn \in MN \). Then \( (mn)x = m(nx) \in MN \) and \( x(mn) = (xm)n \in MN \) since \( nx \in N \) and \( xm \in M \). Thus, \( MN \) is a stable subset of \( X \). Now, suppose that \( x, y \in X \) such that \( x \ast y \in MN \) and \( y \in MN \). Then \( y = mn \) and \( x \ast y = x \ast mn \). Since \( N \) and \( M \) are ideals of \( X \), \( mn \in N \) and \( mn \in M \). Thus, \( x \in M \) and \( x \in N \), so \( x \in MN \). Therefore, \( MN \) is an ideal of \( X \).

Let \( A \) be an ideal of a KS-semigroup \( X \). The relation \( \xrightarrow{A} \) on \( X \) defined by: \( x \xrightarrow{A} y \) if and only if \( x \ast y \in A \) and \( y \ast x \in A \) is an equivalence relation.(see [5])

**Definition 25** Let \( X \) be a KS-semigroup and \( A \) be an ideal of \( X \). Denote \( A_x \) as the equivalence class containing \( x \in X \) and \( X/A \) as the set of all equivalence classes of \( X \) with respect to \( \xrightarrow{A} \), that is, \( A_x = \{y \in X : x \xrightarrow{A} y\} \) and \( X/A = \{A_x : x \in X\} \).

The following results was proved in [5].

**Lemma 26** Let \( X \) be a KS-semigroup and \( A \) be an ideal of \( X \). Then

(i) \( A_x = A_y \) if and only if \( x \xrightarrow{A} y \);

(ii) \( A = A_0 \).

**Theorem 27** If \( A \) is an ideal of a KS-semigroup \( X \), then \( (X/A, \ast, \circ, A_0) \) is a KS-semigroup under the binary operation \( A_x \ast A_y = A_{xy} \) and \( A_x \circ A_y = A_{xy} \) for all \( A_x, A_y \in X/A \).

**Corollary 28** (First Isomorphism Theorem)
If \( f : X \to Y \) is a surjective homomorphism of KS-semigroups, then \( X/Ker f \) is isomorphic to \( Y \).
Parallel to the results in ring theory, this paper proves the second and the third isomorphism theorems for KS-semigroups.

**Theorem 29** (Second Isomorphism Theorem)

Let $M$ and $N$ be ideals of a KS-semigroup $X$ with unity such that $1 \in M \cap N$. Then $M/(M \cap N) \cong MN/N$.

*Proof.* Let $M$, $N$ be ideals of a KS-semigroup $X$. Then $MN$ is an ideal of $X$. Define the function $f : M \rightarrow MN/N$ by $f(m) = Nm$. Let $x, y \in M$ such that $x = y$. Then $x \cdot y = x \cdot x = 0 \in N$ and $y \cdot x = x \cdot x = 0 \in N$. Thus, $x_{\sim N}y$ and $x_{\sim N}y$ by Lemma 26, that is, $f(x) = x_{\sim N}y = f(y)$ and $f$ is well-defined. Also, $f(x \cdot y) = N_{x \cdot y} = x_{\sim N} \cdot y = f(x) \cdot f(y)$ and $f(xy) = N_{xy} = N_x y = f(x)f(y)$. Thus, $f$ is a homomorphism. By FIT, $\text{Im} f \cong M/\text{Ker} f$. To complete the proof, we will show that $\text{Ker} f = M \cap N$ and $\text{Im} f = MN/N$. Let $x \in \text{Ker} f$. Then $N_x = f(x) = N_0$. This implies that $x_{\sim N} 0$. Hence, $x \cdot 0 \in N$. Since $x \in \text{Ker} f \subseteq M$, it follows that $x \in M \cap N$. On the other hand, let $y \in M \cap N$. Then $y \in M$ and $y \in N$. So, $y \cdot 0 = y \in N$ and $0 \cdot y = 0 \in N$. Thus, $y_{\sim N} 0$ and $N_y = N_0$, that is, $f(y) = N_0$ and $y \in \text{Ker} f$. Therefore, $M \cap N \subseteq \text{Ker} f$ and so $M \cap N = \text{Ker} f$. Let $y \in MN/N$. Then $y = N_{mn}$ for some $m \in M$ and $n \in N$. Since $M$ is an ideal of $X$, $mn \in M$ and there exists $m_1 \in M$ such that $m_1 = mn$. Hence, $y = N_{m} = N_{m_1} = f(m_1)$. Therefore, $f$ is onto and so $\text{Im} f = MN/N$. Therefore, $M/(M \cap N) \cong MN/N$. \(\square\)

If $M$ and $N$ are ideals of $X$ such that $M \subseteq N$, then $M$ is an ideal of $N$. So, by Theorem 27, $N/M$ is a sub KS-semigroup of $X/M$.

**Theorem 30** (Third Isomorphism Theorem)

Let $M$ and $N$ be ideals of a KS-semigroup $X$ such that $M \subseteq N$. Then $X/M \cong N/M$.

*Proof.* Let $M$ and $N$ be ideals of a KS-semigroup $X$ such that $M \subseteq N$. Define $f : X/M \rightarrow X/N$ by $f(M_x) = N_x$. Let $M_x, M_y \in X/M$ such that $M_x = M_y$. Then $x_{\sim N} y$. Since $M \subseteq N$, $x_{\sim N} y$. Thus, $f(M_x) = N_x = N_y = f(M_y)$ and $f$ is well-defined. Also, $f(M_x \otimes M_y) = f(M_x \otimes M_y) = N_{x \cdot y} = N_x \otimes N_y = f(M_x) \otimes f(M_y)$ and $f(M_x \otimes M_y) = f(M_x \otimes M_y) = N_{x \cdot y} = N_x \otimes N_y = f(M_x) \otimes f(M_y)$. Therefore, $f$ is a homomorphism. By FIT, $\text{Im} f \cong X/M/\text{Ker} f$. To complete the proof, we will show that $\text{Ker} f = N/M$ and $\text{Im} f = X/N$. Let $M_x \in \text{Ker} f$. Then $N_0 = f(M_x) = N_x$. Thus, $0_{\sim N} x$ and $x = x \cdot 0 \in N$. So, $M_x \in N/M$ and $\text{Ker} f \subseteq N/M$. On the other hand, let $M_x \in N/M$ for some $x \in N$. Then $0 \cdot x = 0 \in M \subseteq N$ and $x = x \cdot 0 \in N$. Thus, $x_{\sim N} 0$. Hence, $N_0 = N_x = f(M_x)$.
and so $M_x \in Kerf$. Thus, $N/M \subseteq Kerf$. So, $N/M = Kerf$. Let $N_x \in X/N$, for some $x \in X$. Then $M_x \in X/M$ and $f(M_x) = N_x$. Thus, $f$ is onto. Therefore, $X/M \cong X/N$. \hfill \Box

The following corollary follows from the proof of Theorem 30 that $N/M = Kerf$.

**Corollary 31** Let $M$ and $N$ be ideals of a KS-semigroup $X$ such that $M \subseteq N$. Then $N/M$ is an ideal of $X/M$.

The following theorem also holds in [5].

**Theorem 32** If $A$ is an ideal of a KS-semigroup $X$, then the mapping $\psi : X \to X/A$ given by $\psi(x) = A_x$ is an epimorphism with kernel $A$.

The preceding theorem now leads to the Correspondence theorem for KS-semigroups.

**Theorem 33** (Correspondence Theorem) If $A$ is an ideal of a KS-semigroup $X$, then there is a one-to-one correspondence between the set of all ideals of $X$ which contain $A$ and the set of all ideals of $X/A$ given by $J \mapsto J/A$. Hence every ideal in $X/A$ is of the form $J/A$ where $J$ is an ideal of $X$ which contains $A$.

**Proof.** Let $A$ be an ideal of $X$. Then by Theorem 32, $f : X \to X/A$ is an epimorphism with kernel $A$. Now, if $J$ is an ideal of $X$, then by Theorem 19(ii), $f(J)$ is an ideal of $X/A$ and if $Y$ is an ideal of $X/A$, then by Theorem 16, $f^{-1}(Y)$ is an ideal of $X$. Thus, the map $\phi$ from the set of all ideals of $X$ which contains $A$ to the set of all ideals of $X/A$ given by $J \mapsto f(J)$ is a well-defined function. Note that if $Y$ is an ideal of $X/A$, then $\{A_o\} = \{A\} \subseteq Y$. Thus, $A = Kerf = f^{-1}(\{A_o\}) \subseteq f^{-1}(Y)$. Hence, $\phi(f^{-1}(Y)) = f(f^{-1}(Y)) = Y$ and $\phi$ is surjective. Next, claim that if $K$ is an ideal of $X$ containing $A$, then $f^{-1}(f(K)) = K$. Clearly, $K \subseteq f^{-1}(f(K))$. Let $x \in f^{-1}(f(K))$. Then $f(x) \in f(K)$, that is, $f(x) = f(k)$ for some $k \in K$. This means that $A_x = A_k$ and so $x * k \in A \subseteq K$. Since $A$ is an ideal, $k \in K$ and $x * k \in K$ imply that $x \in K$. This proves the claim. Now, let $K$ and $L$ be ideals of $X$ containing $A$ such that $\phi(K) = \phi(L)$. Then $f(K) = f(L)$ and $f^{-1}(f(K)) = f^{-1}(f(L))$ which imply that $K = L$ by the claim. Hence, $\phi$ is injective. Finally, observe that $f(J) = J/A$. \hfill \Box
References


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