Inverse Image and Image of Upper and Lower $(\alpha, \beta)$-Fuzzy Set

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Abstract

Using the notion of "belongingness ($\epsilon$)" and "quasi-coincidence ($q$)" of fuzzy points with fuzzy sets. We introduce the concept of inverse image and image of upper and lower of an $(\alpha, \beta)$-fuzzy set, where $\alpha$ and $\beta$ will denote any one of $\epsilon, q, \epsilon \lor q, \epsilon \land q$ with $\alpha \neq \epsilon \land q$, and some interesting properties are investigated.

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1 Introduction

The concept of a fuzzy set in a non-empty set was introduced by Zadeh [12] in 1965. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications (see, for example, [1, 8, 10]).

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [9], played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [3, 4] gave the concepts of $(\alpha, \beta)$-fuzzy subgroups by using the notion of "belongingness ($\epsilon$)" and "quasi-coincidence ($q$)" between a fuzzy point and a fuzzy subgroup, where $\alpha, \beta$ are any two of $\{\epsilon, q, \epsilon \lor q, \epsilon \land q\}$ with $\alpha \neq \epsilon \land q$, and introduced the concept of an $(\epsilon, \epsilon \lor q)$-fuzzy subgroup. In [5] $(\epsilon, \epsilon \lor q)$-fuzzy subrings and ideals defined. In [7] Jun and Song initiated the study of $(\alpha, \beta)$-fuzzy interior ideals of a semigroup. In [2] Bhakat defined $(\epsilon \lor q)$-level subsets of a fuzzy set. In [10] Shabir, Jun et al. studied characterizations of regular semigroups by $(\alpha, \beta)$-fuzzy ideals. In [11] Yuan, Li et al. redefined $(\alpha, \beta)$-intuitionistic fuzzy subgroups. This paper continues this line of research.
2 Preliminaries

Let $X$ be a non-empty set. A mapping $\mu : X \rightarrow [0; 1]$ is called a fuzzy set in $X$. The complement of $\mu$, denoted by $\mu^c$, is the fuzzy set in $X$ given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$. Denote by $FS(X)$ the set of all fuzzy sets in $X$.

For any $t \in [0, 1]$ and fuzzy set $\mu$ of $X$, the set $U(\mu, t) = \{x \in X | \mu(x) \geq t\}$ (respectively, $L(\mu, t) = \{x \in X | \mu(x) \leq t\}$), is called an upper (respectively, lower) $t$-level cut of $\mu$.

**Definition 2.1.** [8] Let $f$ be a mapping from a set $X$ into a set $Y$. Let $\mu$ be a fuzzy set in $X$ and $\lambda$ be a fuzzy set in $Y$. Then the inverse image $f^{-1}(\lambda)$ of $\lambda$ is a fuzzy set in $X$ defined by $f^{-1}(\lambda)(x) = \lambda(f(x))$ for all $x \in X$.

The image $f(\mu)$ of $\mu$ is the fuzzy set in $Y$ defined by

$$ f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} $$

for all $y \in Y$.

We have always $f(f^{-1}(\lambda)) \leq \lambda$ and $\mu \leq f^{-1}(f(\mu))$.

**Definition 2.2.** A fuzzy subset $\mu$ of a universe $X$ is a function from $X$ into the unit closed interval $[0, 1]$, i.e. $\mu : X \rightarrow [0, 1]$ (see [12]). A fuzzy subset $\mu$ in a universe $X$ of the form

$$ \mu(y) = \begin{cases} t \in (0, 1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} $$

is said to be a fuzzy point with support $x$ and value $t$ and is denoted by $x_t$.

For a fuzzy point $x_t$ and a fuzzy set $\mu$ in a set $X$, Pu and Liu [9] gave meaning to the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q, \lor \in, \land \in\}$. A fuzzy point $x_t$ is said to belong to (resp. quasi-coincident with) a fuzzy set $\mu$ written $x_t \in \mu$ (resp. $x_t q \mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, $x_t \in \lor \mu$ (resp. $x_t \in \land \mu$) means that $x_t \in \mu$ or $x_t \lor \mu$ (resp. $x_t \in \mu$ and $x_t \lor \mu$).

In what follows, unless otherwise specified, $\alpha$ and $\beta$ will denote any one of $\in, q, \lor \in$ or $\in \land$ with $\alpha \neq \in \land q$, which was introduced by Bhakat and Das [4]. To say that $x_t \alpha \mu$ means that $x_t \alpha \mu$ does not hold.
3 Inverse Image and Image of Upper and Lower \((\alpha, \beta)\)-Fuzzy Set

**Definition 3.1.** [1] Let \(t \in (0, 1]\) and \(\mu\) is a fuzzy set in \(X\). We defined
\[
\begin{align*}
U(\alpha \mu, t) &= \{x \in X | x \leq \alpha \mu\}, \\
L(\in \mu, t) &= \{x \in X | \mu(x) \leq t\}, \\
L(q \mu, t) &= \{x \in X | \mu(x) + t \leq 1\}, \\
L(\vee q \mu, t) &= \{x \in X | \mu(x) + t \leq 1 \text{ or } \mu(x) \leq t\},
\end{align*}
\]
where \(\alpha \in \{\in, q, \vee q\}\). Then,
\((UL_1)\) the set \(U(\in \mu, t)\) and \(L(\in \mu, t)\) is called an upper and lower \(t\)-level cut of \(\in \mu\), respectively,
\((UL_2)\) the set \(U(q \mu, t)\) and \(L(q \mu, t)\) is called an upper and lower \(t\)-level cut of \(q \mu\), respectively,
\((UL_3)\) the set \(U(\vee q \mu, t)\) and \(L(\vee q \mu, t)\) is called an upper and lower \(t\)-level cut \(\vee q \mu\), respectively.

It is clear that \(U(\in \mu, t) = U(\mu, t)\) and \(L(\in \mu, t) = L(\mu, t)\).

**Theorem 3.2.** [1] Let \(\mu \in FS(X)\). Then for all \(x \in X\) and \(t \in (0, 1]\), we have
\[
(1) \quad U(\in \vee q \mu, t) = U(\in \mu, t) \cup U(q \mu, t), \\
(2) \quad L(\in \vee q \mu, t) = L(\in \mu, t) \cup L(q \mu, t).
\]

**Lemma 3.3.** Let \(\mu \in FS(X)\). Then for all \(x \in X\) and \(t \in (0, 1]\), we have
\[
(1) \quad x_t q \mu \iff x_t \in \mu^c; \\
(2) \quad x_t \in \vee q \mu \iff x_t \in \overline{\vee q \mu}^c.
\]

**Proof.**

(1) Let \(x \in X\) and \(t \in (0, 1]\). Then, we have
\[
\begin{align*}
x_t q \mu & \iff \mu(x) + t > 1 \\
& \iff 1 - \mu(x) < t \\
& \iff \mu^c(x) < t \\
& \iff x_t \in \mu^c.
\end{align*}
\]

(2) Let \(x \in X\) and \(t \in (0, 1]\). Then, we have
\[
\begin{align*}
x_t \in \vee q \mu & \iff x_t \in \mu \text{ or } x_t q \mu \\
& \iff \mu(x) \geq t \text{ or } \mu(x) + t > 1 \\
& \iff 1 - \mu^c(x) \geq t \text{ or } 1 - \mu^c(x) + t > 1 \\
& \iff x_t \overline{\vee q \mu}^c \text{ or } x_t \in \mu^c \\
& \iff x_t \in \overline{\vee q \mu}^c.
\end{align*}
\]
Definition 3.4. A fuzzy set $\mu$ in a set $X$ is said to have sup property if for every non-empty subset $S$ of $X$, there exists $x' \in S$ such that

$$
\mu(x') = \sup_{x \in S} \{\mu(x)\}
$$

Theorem 3.5. Let $\mu, \lambda \in IFS(X)$ and mapping $f$ from $X$ into $Y$ be a surjection. Let $\mu$ and $\lambda$ have sup property, then for all $t \in (0, 1]$ we have

1. $U(\alpha f(\mu), t) = f(U(\alpha \mu, t))$,
2. $L(\alpha f(\lambda), t) \subseteq f(L(\alpha \lambda, t))$,

where $\alpha \in \{\in, q, \in \lor q\}$.

Proof. (1) We only prove the case of $\alpha = q$. The others are analogous.

$y \in U(qf(\mu), t) \iff yqf(\mu)$
$\iff f(\mu)(y) + t > 1$
$\iff \sup_{x \in f^{-1}(y)} \{\mu(x)\} + t > 1$
$\iff \exists x' \in f^{-1}(y), \mu(x') + t > 1$
$\iff f(x') = y, x'q\mu$
$\iff f(x') = y, x' \in U(q\mu, t)$
$\iff y \in f(U(q\mu, t))$.

(2) We only prove the case of $\alpha = \in \lor q$.

Let $y \in L(\in \lor qf(\lambda), t)$. Then, we have $f(\lambda)(y) \leq t$ or $f(\lambda)(y) + t \leq 1$. This shows that

$$
\sup_{x \in f^{-1}(y)} \{\lambda(x)\} \leq t \text{ or } \sup_{x \in f^{-1}(y)} \{\lambda(x)\} + t \leq 1,
$$

and so $\lambda(x) \leq t$ or $\lambda(x) + t \leq 1$ for all $x \in f^{-1}(y)$. This shows that

$$
x \in L(\in \lor q\lambda, t) \text{ for all } x \in f^{-1}(y).
$$

Therefore $y \in f(L(\in \lor q\lambda, t))$.

The other the cases can be proven analogously. \qed

Theorem 3.6. Let $\lambda \in FS(X)$ and mapping $f$ from $X$ into $Y$ be a surjection. Let $\lambda$ and $\lambda^c$ has sup property, then

1. $U(\overline{f}(\lambda), t) = L(qf(\lambda), t) \subseteq U(\in f(\lambda^c), t)$,
(2) \( U(\overline{\alpha f}(\lambda), t) \subseteq L(\in f(\lambda), t) \subseteq U(qf(\lambda^c), t) = L(\overline{\alpha f}(\lambda^c), t) \),

(3) \( L(\overline{\alpha f}(\lambda), t) \subseteq U(\in f(\lambda), t) \),

for all \( t \in (0, 1] \).

**Proof.** (1) Let \( y \in L(qf(\lambda), t) \). Then \( f(\lambda)(y) + t \leq 1 \) and so \( \sup_{x \in f^{-1}(y)} \{ \lambda(x) \} + t \leq 1 \), thus \( \lambda(x) + t \leq 1 \) for all \( x \in f^{-1}(y) \). This shows that \( \lambda^c(x) \geq t \) for all \( x \in f^{-1}(y) \). Then \( \sup_{x \in f^{-1}(y)} \{ \lambda^c(x) \} \geq t \). Since \( \lambda^c \) has sup property, then \( \exists x' \in f^{-1}(y) \), \( \lambda^c(x') \geq t \), and so \( f(x') = y \), \( x'_t \in \lambda^c \). Therefore \( y_t \in f(\lambda^c) \), thus \( y \in U(\in f(\lambda^c), t) \). Hence, we have \( L(qf(\lambda), t) \subseteq U(\in f(\lambda^c), t) \).

Also, \( y \in L(qf(\lambda), t) \) if and only if \( f(\lambda)(y) + t \leq 1 \) if and only if \( y \in U(\overline{\alpha f}(\lambda), t) \). Thus \( U(\overline{\alpha f}(\lambda), t) = L(qf(\lambda), t) \).

The other the cases can be proven analogously. \( \square \)

**Theorem 3.7.** Let \( \mu, \lambda \in FS(X) \) and mapping \( f \) from \( X \) into \( Y \) be a map. Then for all \( t \in (0, 1] \) we have

(1) \( U(\alpha f^{-1}(\mu), t) = f^{-1}(U(\alpha \mu, t)) \),

(2) \( L(\beta f^{-1}(\lambda), t) = f^{-1}(L(\beta \lambda, t)) \),

where \( \alpha \in \{ \in, q \} \) and \( \beta \in \{ \in, q, \wedge q, \in \vee q \} \).

**Proof.** (1) Let \( \alpha = q \). We have

\[
\begin{align*}
x \in U(qf^{-1}(\mu), t) & \iff x, qf^{-1}(\mu) \\
& \iff f^{-1}(\mu)(x) + t > 1 \\
& \iff \mu(f(x)) + t > 1 \\
& \iff f(x), q\mu \\
& \iff f(x) \in U(q\mu, t) \\
& \iff x \in f^{-1}(U(q\mu, t)).
\end{align*}
\]

The other the cases of (1) can be proven analogously.

(2) Let \( \beta = \in \vee q \). We have

\[
\begin{align*}
x \in L(\in \vee qf^{-1}(\lambda), t) & \iff x \in L(\in f^{-1}(\lambda), t) \text{ or } x \in L(qf^{-1}(\lambda), t) \\
& \iff f^{-1}(\lambda)(x) \leq t \text{ or } f^{-1}(\lambda)(x) + t \leq 1 \\
& \iff \lambda(f(x)) \leq t \text{ or } \lambda(f(x)) + t \leq 1 \\
& \iff f(x) \in L(\in \lambda, t) \text{ or } f(x) \in L(q\lambda, t) \\
& \iff f(x) \in L(\in \vee q\lambda, t) \\
& \iff x \in f^{-1}(L(\in \vee q\lambda, t)).
\end{align*}
\]

The other the cases of (2) can be proven analogously. \( \square \)
Theorem 3.8. Let $\mu, \lambda \in FS(X)$ and mapping $f$ from $X$ into $Y$ be a map. Then for all $t \in (0, 1]$ we have

$$L(\in \lor q f^{-1}(\lambda), t) = f^{-1}(L(\in \lambda, t)) \bigcup f^{-1}(L(q \lambda, t)).$$

Proof. By the proof of Theorem 3.7, we have $x \in L(\in \lor q f^{-1}(\lambda), t)$ if and only if $x \in f^{-1}(L(\in \lambda, t))$ or $x \in f^{-1}(L(q \lambda, t))$ if and only if $x \in (f^{-1}(L(\in \lambda, t)) \bigcup f^{-1}(L(q \lambda, t))).$ \hfill \Box

Theorem 3.9. Let $\mu, \lambda \in FS(X)$ and mapping $f$ from $X$ into $Y$ be a map. Then for all $t \in (0, 1]$ we have

$$L(\in \land q f^{-1}(\lambda), t) = f^{-1}(L(\in \lambda, t)) \bigcap f^{-1}(L(q \lambda, t)).$$

Proof. The proof is similar to the proof of Theorem 3.8. \hfill \Box

Theorem 3.10. Let $\lambda \in FS(X)$ and mapping $f$ from $X$ into $Y$ be a map. Then

1. $U(\in f^{-1}(\lambda), t) = L(q f^{-1}(\lambda^c), t),$ 
2. $U(q f^{-1}(\lambda), t) = L(\in f^{-1}(\lambda^c), t),$ 

for all $t \in (0, 1].$

Proof. (1) Let $x \in L(q f^{-1}(\lambda^c), t).$ Then, if and only if $f^{-1}(\lambda^c)(x) + t \leq 1$ if and only if $\lambda^c(f(x)) + t \leq 1$ if and only if $\lambda(f(x)) \geq t$ if and only if $f^{-1}(\lambda)(x) \geq t$ if and only if $x \in U(\in f^{-1}(\lambda), t).$ Hence, we have $U(\in f^{-1}(\lambda), t) = L(q f^{-1}(\lambda^c), t).$

(2) The proof is similar to the proof of (1). \hfill \Box

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References

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