Generalized Matrix Near Ring 
over Abstract Affine Near Ring

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Abstract. Let $A$ be an abstract affine near ring, $M$ be a faithful near ring $A$–module and $n$ be a positive integer. In this paper we define the $n \times n$ generalized matrix near ring over $A$ using the faithful near ring $A$–module $M$ which is denoted by $\text{Mat}_n(A, M)$. Also we find a necessary and sufficient condition for which $\text{Mat}_n(A, M)$ is an abstract affine near ring.

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1. Introduction

The theory of near-rings is presented in[3]. We recall some concepts of this theory. Let $A = (A, +, \cdot)$ be an abstract affine near-ring (a.a.n.r for short ), i.e. $(A, +)$ is an abelian group, $(A, \cdot)$ is a semigroup, $(a + b) \cdot c = a \cdot c + b \cdot c$, for all $a, b, c \in A$, and $A_0 = A_d$, where $A_0 = \{a \in A : a \cdot 0 = 0\}$ is the zero symmetric part and $A_d = \{a \in A : a \cdot (x + y) = a \cdot x + a \cdot y, \text{for all } x, y \in A\}$ is the distributive part. Let $R$ be a ring and $M$ a left $R$–module. By [3, Prop.9.81] there is exactly one way to extend the multiplication $\cdot : R \times M \to M$ to a multiplication ”$\circ$” in $(A, +) = (R, +) \oplus (M, +)$ such that $(A, +, \circ)$ is a near-ring with $A_d = A_0 = R \oplus (0)$ and $A_c = (0) \oplus M$, namely $(r, m) \circ (s, n) = (r \cdot s, r \cdot n + m)$. Moreover, $A = (A, +, \circ)$ is an aann and all aann’s arise in this way. This aann will be denoted by $R \ast M$.

Conversely, for any aann $A$, the zero symmetric part $A_0$ is a ring and the constant part $A_c = \{a \in A : a \cdot x = a, \text{for all } x \in A\}$ is a left module over $A_0$.

Let $A$ be an abstract affine near ring, $N$ be a faithful near ring $A$–module and $n$ be a positive integer. In this paper we define the $n \times n$ generalized matrix near ring over $A$ using the faithful near ring $A$–module $N$ which is denoted by $\text{Mat}_n(A, M)$ as the direct sum of the two near rings $\text{Mat}_n(A_0, N)$ and $\text{Mat}_n(A_d, M)$. 


and $A^n_c$, where $\text{Mat}_n(A_0, N)$ is the generalized matrix near ring as in [2], $A^n_c$ is the direct sum of $n$ copies of $A_c$ which is a near ring under component wise addition and multiplication.

It is clearly that the direct sum of two right near rings $S, D$ which is denoted by $S \oplus D$ is a right near ring under component wise addition and multiplication.

In the following $\oplus$ means direct sum, this lemma is important in our work

**Lemma 1.1**

Let $S_1$ and $D_1$ are right zero symmetric near rings and they are isomorphic (i.e $S_1 \cong D_1$), $S_2$ and $D_2$ are right constant near rings and isomorphic (i.e $S_2 \cong D_2$), then $S = S_1 \oplus S_2, D = D_1 \oplus D_2$ are isomorphic (i.e $S \cong D$).

**Proof**

Let $f_1 : S_1 \to D_1, f_2 : S_2 \to D_2$ are isomorphisms. Define $f : S \to D$, by $f(s_1, s_2) = (f_1(s_1), f_2(s_2))$.

It is clearly that $f$ is an isomorphism.

From [3] if $A = R \ast M$ be an abstract affine near-ring and $n > 1$ be a natural number. Then $M_n(R \ast M) \cong M_n(R) \ast M^n$.

In this paper we extend this to define the generalized matrix near ring $M_n(A, M)$ when $A$ is an abstract affine near ring.

2. Notations and Definitions

If $A$ is a right near ring with identity, $n$ be any positive integer. In 1986 J.D.P. Meldrum and A.P.J. Van der Walt define Matrix near ring over $A$, $\text{Mat}_n(A)$, regards $A$ as a left module over $A$. In [2] $A$ denote a right, zero symmetric near ring with identity, $M$ be a faithful left $A$–module, $(M, +)$ need not be a belian group and $M^n$ is the direct sum of $n$ copies of $M$, also a faithful left $A$–module.

Kirby C. Smith in [2] define the $n \times n$ generalized matrix near ring over $A$ using the faithful left $A$–module $M$ as the subnear ring $M_n(A, M)$ of $M_0(M^n)$ generated by $f^r_{ij}, r \in R$ and $1 \leq i, j \leq n$, where the generalized $n \times n$ matrix near ring will be a function from $M^n$ to $M^n$. Now we define special functions in $M_0(M^n)$ will be denoted by $f^r_{ij}, r \in A$ and $1 \leq i, j \leq n$

$$f^r_{ij} : M^n \to M^n \text{ such that } r \in A \text{ and } 1 \leq i, j \leq n,$$

$$f^r_{ij}(a_1, ..., a_k) = (0, ..., 0, ra_j, 0, ..., 0) \text{ where } ra_j \text{ in the } i-th \text{ position},$$

$$(a_1, ..., a_k) \in M^n \text{ and } f^r_{ij} = l_i f^r_{ij} \pi_j \text{ where } l_i : M \to M^n \text{is the } i-th \text{ injection, }\pi_j : M^n \to M \text{ is the } j-th \text{ projection and } f^r : M \to M \text{ such that } f^r(s) = rs \forall s \in M,$$

So $f^r_{ij}$ is the function from $M^n$ to $M^n$ that takes a $n$–tuple with entries from $M$, multiplies the $j$–th entry $a_j$ by $r$ using the module action of $R$ on $M$, puts the result $ra_j$ into the $i$–th position and puts 0 in the other positions. We may sometimes write $f^r_{ij}$ as $[r; i, j]$. 
3. Main Results

**Definition 3.1**
Let $A$ be any abstract affine near ring, $N$ be a faithful $A$--module then generalized matrix near ring with respect to $A$ and $N$ which is denoted by $\text{Mat}_n(A, N)$ is the direct sum of $\text{Mat}_n(A_0, N)$ and $A^n_c$ i.e

$$\text{Mat}_n(A, N) = \text{Mat}_n(A_0, N) \oplus A^n_c$$

In the following $A = A_0 \ast A_c$ be any abstract affine near ring, $N, M$ are faithful $A$--modules, $n$ be a positive integer

**Lemma 3.2**

$$\text{Mat}_1(A, N) \cong A \cong \text{Mat}_1(A)$$

**Proof**
We have

$$\text{Mat}_1(A, N) = \text{Mat}_1(A_0, N) \oplus A_c$$

but also we have

$$\text{Mat}_1(A_0, N) \cong \text{Mat}_1(A_0) \cong A_0$$

and

$$A_c \cong \text{Mat}_1(A_c).$$

So by lemma 1.1

$$\text{Mat}_1(A, N) = \text{Mat}_1(A_0, N) \oplus A_c \cong A_0 \oplus A_c = A$$

$$\cong \text{Mat}_1(A_0) \oplus \text{Mat}_1(A_c) = \text{Mat}_1(A).$$

**Theorem 3.3**
If $\theta : M \rightarrow N$ is an $A$--epimorphism, then $\theta$ induces a near ring epimorphism from $\text{Mat}_n(A; M)$ into $\text{Mat}_n(A; N)$. So if $N$ is a homomorphic image of $M$, then the matrix near ring $\text{Mat}_n(A; N)$ is a homomorphic image of $\text{Mat}_n(A; M)$.

**Proof**
Since $M, N$ are faithful $A$--modules so $M, N$ are faithful $A_0$--modules and so we have

$$\text{Mat}_n(A, M) = \text{Mat}_n(A_0, M) \oplus A^n_c$$

$$\text{Mat}_n(A, N) = \text{Mat}_n(A_0, N) \oplus A^n_c.$$

But from [2, Theorem1] we have $\text{Mat}_n(A_0; N)$ is a homomorphic image of $\text{Mat}_n(A_0; M)$.

So

$$\text{Mat}_n(A, M) = \text{Mat}_n(A_0, M) \oplus A^n_c$$

is a homomorphic image of $\text{Mat}_n(A, N) = \text{Mat}_n(A_0, N) \oplus A^n_c$.

**Corollary 3.4**
Let $M$ be a homomorphic image of $A A$. Then $\text{Mat}_n(A; M)$ is isomorphic to $\text{Mat}_n(A; A)$ for all $n \geq 1$.

**Proof**
Since $M$ is a faithful $A$--module which is a homomorphic image of $A A$ so $M$
be a faithful $A_0$—module which is a homomorphic image of $A_0 A_0$. Then from [2,Corollary1]
$$M_n(A_0, M) \cong M_n(A_0, A_0)$$
and so
$$M_n(A; M) = M_n(A_0, M) \oplus A^n_c$$
$$\cong M_n(A_0, A_0) \oplus A^n_c$$
$$\cong M_n(A_0) \oplus A^n_c$$
$$\cong M_n(A)$$

**Corollary 3.5**
If $M$ and $N$ are isomorphic faithful $A$—modules, then
$$M_n(A; M) \simeq M_n(A; N)$$
for all $n \geq 1$.

**Proof**
Since $M$ and $N$ are isomorphic faithful $A$—modules, then $M$ and $N$ are isomorphic faithful $A_0$—modules and so from [2,Corollary2]
$$M_n(A_0; M) \simeq M_n(A_0; N)$$
for all $n \geq 1$.

Now for all $n \geq 1$
$$M_n(A; M) = M_n(A_0; M) \oplus A^n_c$$
$$\cong M_n(A_0; N) \oplus A^n_c$$
$$= M_n(A_0; N).$$

**Lemma 3.6**
$$M_n(A_0, A) \cong M_n(A_0, A_0)$$

**Proof**
From [2, Corollary1] since $A$ is a modul over $A_0$ under the usual multiplication and is a homomorphic image of the module $A_0$ over $A_0$.

**Proposition 3.7**
$$M_n(A, A) \cong M_n(A)$$

**Proof**
$$M_n(A, A) = M_n(A_0, A) \oplus A^n_c$$
$$\cong M_n(A_0, A_0) \oplus A^n_c$$
$$\cong M_n(A_0) \oplus A^n_c$$
$$\cong M_n(A)$$
Lemma 3.8

\[(M_n(A, M))_d \cong M_n(A_d, M)\]

Proof

Since \(A_d\) is embeded in \(A\) then \(M_n(A_d, M)\) is embeded in \(M_n(A, M)\) i.e

\[M_n(A_d, M) \hookrightarrow M_n(A, M)\]

so

\[(M_n(A_d, M))_d \hookrightarrow (M_n(A, M))_d .\]

But \(M_n(A_d, M)\) is distributive so \((M_n(A_d, M))_d = M_n(A_d, M)\) which implies that

\[M_n(A_d, M) \hookrightarrow (M_n(A, M))_d .\]

But there is a one to one correspondence between the set of generators of \((\text{Mat}_n(A, M))_d\) and the set of generators of \(\text{Mat}_n(A_d, M)\) because \((M_n(A, M))_d\) is embeded in the the zero-symmetric part of \(M_n(A, M)\) and we have

\[(f_{ij}^r, 0) \in (M_n(A, M))_d \text{ iff } r \in A_d.\]

So

\[(M_n(A, M))_d \cong \text{Mat}_n(A_d, M).\]

We use lemma 3.8 to proof the following theorem which gives a necessary and sufficient condition for which \(\text{Mat}_n(A, M)\) is an abstract affine near ring.

Theorem 3.9

\(M_n(A_0*A_c, N)\) is abstract affine near ring (a.a.n.r for short ) iff \(N\) is a ring \(A_0\)–module

Proof

\((\Rightarrow)\) Let \(M_n(A_0 * A_c, N)\) is an a.a.n.r then we have \(M_n(A_0 * A_c, N)\) is abelian and so \(N\) is abelian (1). We show that

\[\forall r \in A_0, \forall l, m \in N \quad r(l + m) = rl + rm.\]

Since\(M_n(A_0*A_c, N)\) is an a.a.n.r then \((M_n(A_0*A_c, N))_0 = (\text{Mat}_n(A_0*A_c, N))_d\)

and so we have if \(r \in A_0, 1 \leq i, j, k, h \leq n\)

\[(f_{ij}^r, 0)((f_{jk}^1, 0) + (f_{jh}^1, 0)) = (f_{ij}^r, 0)(f_{jk}^1, 0) + (f_{ij}^r, 0)(f_{jh}^1, 0)\]

so if \(n_k = m, n_h = l \in N\) then

\[(f_{ij}^r, 0)((f_{jk}^1, 0) + (f_{jh}^1, 0))(0, ..., m, 0, ..., l, 0, ..., 0) =\]

\[(f_{ij}^r, 0)(f_{jk}^1, 0) + (f_{ij}^r, 0)(f_{jh}^1, 0))(0, ..., m, 0, ..., l, 0, ..., 0)\]

so

\[(f_{ij}^r, 0)(0, ..., m + l, 0, ..., 0) = (f_{ij}^r, 0)(0, ..., m, 0, ..., 0) + (f_{ij}^r, 0)(0, ..., l, 0, ..., 0)\]
then
\[ r(m + l) = rm + rl \quad (2) \]
so from (1),(2) we have \( N \) is a ring \( A_0 \)--module.

\((\Leftarrow)\) Let \( N \) be a ring \( A_0 \)--module we have \( M_n(A_0 \ast A_c, N) \) is the direct sum of \( M_n(A_0 \oplus \{0\}, N), A^n_c \) i.e
\[ M_n(A_0 \ast A_c, N) = M_n(A_0, N) \oplus A^n_c. \]
Since \( A_0, N \) are abelian so \( M_n(A_0, N) \) is abelian also \( A^n_c \) is abelian since \( A_c \) is a belian then \( M_n(A_0 \ast A_c, N) \) is abelian \((1)\). Also from lemma 3.8 we have
\[ (M_n(A_0 \ast A_c, N))_d \cong M_n((A_0 \ast A_c)_d, N)) = M_n(A_0, N) \quad (2) \]
\( M_n(A_0 \ast A_c, N) \) is a.a.n.r.

**REFERENCES**


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