Orthogonal Derivations on an Ideal
of Semiprime Γ-Rings

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Abstract

In this paper, we generalized some results concerning orthogonal derivations for a nonzero ideal of a semiprime Γ-ring. These results which are related to some results concerning product derivations on a Γ-rings.

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1. Introduction

Nobusawa [5] introduced the notion of a Γ-ring, more general than a ring. Burnes [2] weakened the conditions in the definition of Γ-ring in the sense of Nobusawa. After these two authors, many mathematicians made works on Γ-ring in the sense of Barnes and Nobusawa, which are parallel to the results in the ring theory.

The gamma ring is defined by Barnes in [2] as follows:
A Γ-ring is a pair \((M, \Gamma)\) where \(M\) and \(\Gamma\) are additive abelian groups for which there exists a map from \(M \times \Gamma \times M \to M\) (the image of \((x, \alpha, y)\) was denoted by \(x\alpha y\)) for all \(x, y, z \in M\) and \(\alpha \in \Gamma\) satisfying the following conditions:

(i) \(x\alpha y \in M\)
(ii) \((x + y)\alpha z = x\alpha z + y\alpha z\),
\[ x(\alpha + \beta)y = x\alpha y + x\beta y, \]
\[ xa(y + z) = x\alpha y + x\alpha z, \]
(iii) \((x\alpha y)\beta z = x\alpha (y\beta z)\).

We may note that it follows from (i)→(iii) that \(0\alpha x = x0y = 0\alpha x = 0\), for all \(x, y \in M\) and \(\alpha \in \Gamma\).

A Γ-ring \(M\) is said to be 2-torsion free if \(2x = 0\) implies \(x = 0\) for \(x \in M\). \(M\) is called a prime if for any two elements \(x, y \in M\), \(x\Gamma M \subseteq y \Gamma M \subseteq M\) implies \(x = 0\) or \(y = 0\), and \(M\) is called semiprime if \(x\Gamma M \subseteq x = 0\) with \(x \in M\) implies \(x = 0\). Note that every prime Γ-ring is obviously semiprime. An additive subgroup \(U\) of \(M\) is called a left (right) ideal of \(M\) if \(M \Gamma U \subseteq U\) (\(U \Gamma M \subseteq U\)). If \(U\) is both left and right ideal of \(M\), then we say \(U\) is an ideal of \(M\). Following [6] a subset \(U\) of \(M\), \(\text{Ann}_\Gamma(U) = \{a \in M\ | a\Gamma U = \langle 0 \rangle \}\) is called the left annihilator of \(U\). A right annihilator \(\text{Ann}_r(U)\) can be defined similarly. It is known that the right and left annihilators of an ideal \(U\) of a semiprime Γ-ring \(M\) coincide, it will be denoted by \(\text{Ann}(U)\). Not that \(U \cap \text{Ann}(U) = \{0\}\) \((U \cap \text{Ann}_r(U) = \{0\})\). Jing in [4], defined the derivation of Γ-ring as follows: An additive mapping \(d: M \to M\) is called a derivation if \(d(x\alpha y) = d(x)\alpha y + x\alpha d(y)\) for all \(x, y \in M\) and \(\alpha \in \Gamma\).

Beršar and Vukman in [3] introduced the notion of orthogonality for a pair of derivations \((d, g)\) of a semiprime ring, and they gave several necessary and sufficient conditions for \(d, g\) to be orthogonal on a semiprime Γ-ring. Ashraf and Jamal in [1] they study the concepts of orthogonal derivation in Γ-ring \(M\) as follows: Two mappings \(f\) and \(g\) of a Γ-ring \(M\) are said to be orthogonal on \(M\) if

\[ f(x)\Gamma M \Gamma g(y) = 0 = g(y)\Gamma M \Gamma f(x) \]
for all \(x, y \in M\) and \(\alpha \in \Gamma\),

and obtained some results analogous to obtained by Beršar and Vukman [3].

In this paper we extend the results Ashraf and Jamal in [1] to orthogonal derivation on a nonzero ideal of semiprime Γ-ring \(M\) and \(M\) satisfying \(x\alpha y\beta z = x\beta y\alpha z\), for all \(x, y, z \in M\) and \(\alpha, \beta \in \Gamma\), and it will be represented by \((*)\).
2. The Results

To prove the main result we need the following lemmas.

**Lemma 2.1.** Let $M$ be a 2-torsion free semiprime $\Gamma$-ring, $U$ a nonzero ideal of $M$ and $a, b$ the elements of $M$. Then the following conditions are equivalent:

(i) $a\Gamma U \Gamma b = (0)$

(ii) $b\Gamma U \Gamma a = (0)$

(iii) $a\Gamma U \Gamma b + b\Gamma U \Gamma a = (0)$

If one of these conditions are satisfying and Ann$_l(U) = 0$, then $a\Gamma b = b\Gamma a = 0$.

**Proof.** (i)$\rightarrow$(ii) Suppose that $a\Gamma U \Gamma b = 0$. Then

$b\Gamma U \Gamma a = 0$, since $U$ is an ideal then $b\Gamma U \Gamma a \Gamma U = 0$.

By semiprimeness, $b\Gamma U \Gamma a = 0$, hence $b\Gamma U \Gamma a \in \text{Ann}_l(U) = 0$, we get $b\Gamma U \Gamma a = 0$.

(ii)$\rightarrow$(iii) Suppose that $b\Gamma U \Gamma a = 0$, that is $a\Gamma U \Gamma b = 0$, this implies $a\Gamma U \Gamma b + b\Gamma U \Gamma a = 0$.

(iii)$\rightarrow$(i) Suppose that $a\Gamma U \Gamma b + b\Gamma U \Gamma a = 0$, that is $a\Gamma U \Gamma b = -b\Gamma U \Gamma a$.

Let $u$ and $v$ be any two elements of $U$. Then by hypotheses we have

$$(a\Gamma u \Gamma b) \Gamma v \Gamma (a\Gamma u \Gamma b) = - a\Gamma u \Gamma a \Gamma v \Gamma b \Gamma u \Gamma b$$

$$= a\Gamma u \Gamma b \Gamma v \Gamma (b \Gamma u \Gamma a)$$

$$= - a\Gamma u \Gamma b \Gamma v \Gamma a \Gamma u \Gamma b$$

This implies $2(a\Gamma u \Gamma b) \Gamma v \Gamma (a\Gamma u \Gamma b) = 0$.

Since $M$ 2-torsion free $\Gamma$-ring, we obtain $(a\Gamma u \Gamma b) \Gamma v \Gamma (a\Gamma u \Gamma b) = 0$.

Since $U$ be an ideal, then $(a\Gamma u \Gamma b) \Gamma U = 0$. By the semiprimeness we get $(a\Gamma u \Gamma b) \Gamma U = 0$, hence $a\Gamma u \Gamma b \in \text{Ann}_l(U) = 0$, $a\Gamma u \Gamma b = 0$. For all $u \in U$. Hence we get $a\Gamma u \Gamma b = b\Gamma u \Gamma a = 0$.

**Lemma 2.2.** Let $M$ be a 2-torsion free semiprime $\Gamma$-ring, and $U$ be a nonzero ideal of $M$ such that Ann$_l(U) = 0$. Suppose that additive mappings $f$ and $h$ of $M$ into itself satisfy $f(x)\Gamma U \Gamma h(x) = (0)$ for all $x \in U$. Then $f(x)\Gamma U \Gamma h(y) = (0)$ for all $x, y \in U$.

**Proof.** Suppose that $f(x)\alpha \Gamma U \Gamma h(x) = 0$ for all $x, u \in U$ and $\alpha, \beta \in \Gamma$. Linearizing we get

$$f(x)\alpha \Gamma U \Gamma h(x) + f(y)\alpha \Gamma U \Gamma h(x) = 0$$

for all $x, y, u \in U$ and $\alpha, \beta \in \Gamma$.

Then we have
Replacing \( v \) by \( v_{\tau} \), we get

\[
f(x)^{\alpha}u^{\beta}h(y)^{\gamma}v^{\tau}m = 0,
\]
for all \( x, y, u, v \in U \) and \( \alpha, \beta, \gamma, \delta, \tau \in \Gamma \).

By semiprimeness we obtain \( f(x)^{\alpha}u^{\beta}h(y)^{\gamma}v = 0 \), that is \( f(x)^{\alpha}u^{\beta}h(y) \in \text{Ann}(U) = 0 \), this implies \( f(x)^{\alpha}u^{\beta}h(y) = 0 \), for all \( x, y, u \in U \) and \( \alpha, \beta \in \Gamma \).

**Lemma 2.3.** Let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring, \( d \) and \( g \) be derivations of \( M \), and \( U \) be a nonzero ideal of \( M \) such that \( \text{Ann}(U) = 0 \). If \( d(x)^{\alpha}g(y) + g(x)^{\alpha}d(y) = 0 \) for all \( x, y \in U \) and \( \alpha \in \Gamma \), then \( d \) and \( g \) are orthogonal.

**Proof.** Suppose that \( d(x)^{\alpha}g(y) + g(x)^{\alpha}d(y) = 0 \) for all \( x, y \in U \) and \( \alpha \in \Gamma \). Replacing \( y \) by \( y^{\beta}m \) we get

\[
0 = d(x)^{\alpha}g(y^{\beta}m) + g(x)^{\alpha}d(y^{\beta}m)
\]
\[
= d(x)^{\alpha}g(y)^{\beta}m + d(x)^{\alpha}y^{\beta}g(m) + g(x)^{\alpha}d(y)^{\beta}m + g(x)^{\alpha}y^{\beta}d(m)
\]
\[
= d(x)^{\alpha}y^{\beta}g(m) + g(x)^{\alpha}y^{\beta}d(m), \text{ for all } x, y \in U, m \in M \text{ and } \alpha \in \Gamma.
\]

Replacing \( x \) by \( mx \) we get

\[
0 = d(mx)^{\alpha}y^{\beta}g(m) + g(mx)^{\alpha}y^{\beta}d(m)
\]
\[
= d(mx)^{\alpha}y^{\beta}g(m) + m^{\gamma}d(x)^{\alpha}y^{\beta}g(m) + g(m)^{\gamma}x^{\alpha}y^{\beta}d(m) + m^{\gamma}g(x)^{\alpha}y^{\beta}d(m)
\]
\[
= d(mx)^{\alpha}y^{\beta}g(m) + g(m)^{\gamma}x^{\alpha}y^{\beta}d(m), \text{ for all } x, y \in U, m \in M \text{ and } \alpha, \beta, \gamma \in \Gamma.
\]

Since \( x^{\alpha}y^{\beta}g(m) \in U \). Put \( u = x^{\alpha}y^{\beta}g(m) \), then by Lemma 2.1 we get \( d(m)^{\gamma}u^{\beta}g(m) = 0 \). By Lemma 2.2 yield \( d(m)^{\gamma}u^{\beta}g(s) = 0 \), for all \( u \in U, m, s \in M \) and \( \beta, \gamma \in \Gamma \). Replacing \( u \) by \( t \), we have \( d(m)^{\gamma}t^{\beta}g(s) = 0 \). Replacing \( u \) by \( ur \) we have \( d(m)^{\gamma}t^{\beta}g(s) = 0 \). By semiprimeness and (*) we get \( d(m)^{\gamma}t^{\beta}g(s) = 0 \), for all \( m, s, t \in M \) and \( \gamma \in \Gamma \). Hence \( d \) and \( g \) are orthogonal.

**Theorem 2.4.** [1, Theorem 2.1] Let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring. Suppose \( d \) and \( g \) are derivations of \( M \). Then the following conditions are equivalent:

(i) \( d \) and \( g \) are orthogonal.
(ii) \( dg = 0 \).
(iii) \( dg + gd = 0 \).
(iv) \( \text{dg} \) is a derivation.
(v) there exists \( a, b \in M \) and \( \alpha, \beta \in \Gamma \) such that \( (\text{dg})(x) = a\beta x + x\gamma b \).

Now we prove the main result.

**Theorem 2.5.** Let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring, \( d \) and \( g \) are derivations of \( M \), and \( U \) be a nonzero ideal of \( M \) such that \( \text{Ann}_l(U) = 0 \). Then the following conditions are equivalent:

(i) \( d \) and \( g \) are orthogonal of \( M \).
(ii) \( dg = 0 \) on \( U \).
(iii) \( dg + gd = 0 \) on \( U \).
(iv) \( \text{dg} \) is a derivation on \( U \).
(v) there exists \( a, b \in M \) and \( \alpha, \beta \in \Gamma \) such that \( (\text{dg})(x) = a\beta x + x\gamma b \), for all \( x \in U \).

**Proof.** (i) \( \rightarrow \) (ii), (iii), (iv)and (v) are clear by Theorem 2.4.

(ii)\( \rightarrow \) (i) Suppose that \( \text{dg} = 0 \) on \( U \), that is \( \text{dg}(x) = 0 \), for all \( x \in U \).
Replacing \( x \) by \( x\alpha m \), we get

\[
0 = d(g(x)\alpha m + x\alpha g(m))
\]
\[
= d(g(x))\alpha m + g(x)d(\alpha m) + d(x)\alpha g(m) + x\alpha d(g(m))
\]
\[
= g(x)d(\alpha m) + d(x)\alpha g(m) + x\alpha d(g(m))
\]
Replacing \( x \) by \( m\beta x \), we get

\[
0 = g(m)\beta x\alpha d(m) + m\beta g(x)\alpha d(m) + d(m)\beta x\alpha g(m) + m\beta d(x)\alpha g(m) + m\beta x\alpha d(g(m))
\]
\[
= g(m)\beta x\alpha d(m) + d(m)\beta x\alpha g(m)
\]

By Lemma 2.1 we have \( g(m)\beta x\alpha d(m) = 0 \). By Lemma 2.2 we get \( g(m)\beta x\alpha d(s) = 0 \), for all \( x \in U \), \( m, s \in M \) and \( \alpha, \beta \in \Gamma \). Replacing \( x \) by \( t\alpha d(s) \gamma x \alpha \), we get

\[
g(m)\beta t\alpha d(s)\gamma x \alpha \delta g(m)\beta t\alpha d(s) = 0
\]

By semiprimeness we obtain \( g(m)\beta t\alpha d(s)\gamma x = 0 \), for all \( x \in U \), \( m, s, t \in M \) and \( \alpha, \beta, \gamma, \delta, \lambda \in \Gamma \). Hence \( d \) and \( g \) are orthogonal.

(iii) \( \rightarrow \) (i) Suppose that \( \text{dg} + gd = 0 \), that is \( (\text{dg} + gd)(x) = 0 \) for all \( x \in U \).
Replacing \( x \) by \( x\alpha m \), we obtain
\[ 0 = d(g(x)am + xag(m)) + g(d(x)am + xad(m)) \]
\[ = d(g(x))am + d(x)ag(m) + xad(g(m)) + g(d(x))am + d(x)ag(m) + g(x)ad(m) + xag(d(m)) \]
\[ = 2d(x)ag(m) + 2g(x)ad(m) + x\{d(g(m)) + g(d(m))\} \]

Replacing \( x \) by \( m^\beta x \) we get
\[ 0 = 2\{d(m^\beta x)ag(m) + m^\beta d(x)ag(m) + g(m^\beta x)ad(m) + m^\beta g(x)ad(m)\} + m^\beta x\{d(g(m)) + g(d(m))\} \]
\[ = 2\{d(m)\beta xag(m) + g(m)\beta xad(m)\} \]

Since \( M \) is \( 2 \)-torsion free \( \Gamma \)-ring, we have
\[ d(m)\beta xag(m) + g(m)\beta xad(m) = 0, \text{ for all } x \in U, m \in M \text{ and } \alpha, \beta \in \Gamma. \]

By Lemma 2.1 we get \( d(m)\beta xag(m) = 0 \). By Lemma 2.2 we get \( d(m)\beta xag(s) = 0 \), for all \( m, s \in M \) and \( \alpha, \beta \in \Gamma \). Replacing \( x \) by \( t^\gamma x^\delta d(m)\beta t \) we have
\[ d(m)\beta t\gamma x^\delta d(m)\beta x\delta (m)\beta t\gamma d(s) = 0. \]
Replacing \( x \) by \( x^\lambda r \), we get
\[ d(m)\beta t\gamma x^\delta d(m)\beta t\gamma d(s) = 0, \text{ for all } x \in U, m, s \in M \text{ and } \alpha, \beta, \gamma, \delta, \lambda \in \Gamma. \]

By semiprimeness we obtain \( d(m)\beta t\gamma x^\delta d(m)\beta t\gamma d(s) = 0, \text{ for all } m, s, t \in M \text{ and } \alpha, \beta \in \Gamma. \) Hence \( d \) and \( g \) are orthogonal.

(iv) \( \leftrightarrow \) (i) Suppose that \( dg \) is derivation from \( U \) to \( M \), we have
\[ dg(xay) = d(g(x))ay + x^\beta d(g(y)), \text{ for all } x, y \in U \text{ and } \alpha \in \Gamma. \]  
(2.1)

In other hand
\[ dg(xay) = d(g(x))ay + g(x)ad(y) + d(x)ag(y) + x^\beta d(g(y)) \]  
(2.2)

Comparing (2.1) and (2.2) we get
\[ d(x)ag(y) + g(x)ad(y) = 0, \text{ for all } x, y \in U \text{ and } \alpha \in \Gamma \]

Hence by Lemma 2.3 we get \( d \) and \( g \) are orthogonal.

(v) \( \rightarrow \) (i) Suppose that there exists \( a, b \in M \) and \( \beta, \gamma \in \Gamma \) such that \( dg(x) = a^\beta x + x^\gamma b \). Replacing \( x \) by \( xam \) we get
\[ dg(xam) = d(g(x)am + xag(m)) \]
Orthogonal derivations on ideal

\[ a\beta x\alpha m + x\alpha m\gamma b = dg(x)\alpha m + d(x)\alpha g(m) + x\alpha dg(m) \]
\[ a\beta x\alpha m + x\alpha m\gamma b = a\beta x\alpha m + xybam + g(x)\alpha d(m) + d(x)ag(m) + x\alpha dg(m), \] that is
\[ x\gamma bam + g(x)\alpha d(m) + d(x)\alpha g(m) + x\alpha dg(m) \]
\[ - x\alpha m\gamma b = 0. \]

Replacing \( x \) by \( m\delta x \) we get
\[ m\delta x\gamma bam + g(m)\delta x\alpha d(m) + m\delta g(x)\alpha d(m) + m\delta x\alpha dg(m) +
\[ m\delta x\alpha dg(m) - m\delta x\alpha m\gamma b = 0. \]

Therefore
\[ g(m)\delta x\alpha d(m) + d(m)\delta x\alpha g(m) = 0, \] for all \( x \in U \), \( m \in M \) and \( \alpha, \delta \in \Gamma \).

By Lemma 2.1 we have \( g(m)\beta x\alpha d(m) = 0 \). By Lemma 2.2 we get \( g(m)\beta x\alpha d(s) = 0 \), for all \( x \in U \), \( m, s \in M \) and \( \alpha, \beta \in \Gamma \). Replacing \( x \) by \( t\alpha d(s)\gamma x\delta g(m)\beta t \) we have
\[ g(m)\beta t\alpha d(s)\gamma x\delta g(m)\beta t d(s) = 0. \]
Replacing \( x \) by \( x\alpha r \), we get
\[ g(m)\beta t\alpha d(s)\gamma x\alpha r\delta g(m)\beta t d(s)\gamma x = 0, \] for all \( x \in U \), \( m, s, t \in M \) and \( \alpha, \beta, \gamma, \delta, \lambda \in \Gamma \).

By semiprimeness we obtain \( g(m)\beta t\alpha d(s)\gamma x = 0 \), yields \( g(m)\beta t\alpha d(s) \in Ann_l(U) = 0 \), therefore \( g(m)\beta t\alpha d(s) = 0, \) for all \( m, s, t \in M \) and \( \alpha, \beta \in \Gamma \). Hence \( d \) and \( g \) are orthogonal.

**Corollary 2.6.** Let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring, \( d \) and \( g \) are derivations from \( U \) to \( M \), and \( U \) be a nonzero ideal such that \( Ann_l(U) = 0 \). If \( dg \) is derivation from \( U \) to \( M \). Then \( dg \) is derivation of \( M \).

**Corollary 2.7.** Let \( M \) be a 2-torsion free semiprime \( \Gamma \)-ring, \( d \) be a derivation from \( U \) to \( M \), and \( U \) be a nonzero ideal such that \( Ann_l(U) = 0 \). If \( d^2 \) is derivation, then \( d = 0 \).

**Proof.** Suppose that \( d^2 \) is derivation on \( U \), by Theorem 2.5 we get \( d \) and \( d \) are orthogonal. That is \( d(x)\Gamma M\Gamma d(x) = 0 \). Then by semiprimeness we get \( d = 0 \).

**References**


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