Tripled Coincidence Points for Monotone Operators in Partially Ordered Metric Spaces

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Abstract. Using the notion of compatible mappings in the setting of a partially ordered metric space, we prove the existence and uniqueness of tripled coincidence points involving a \((\phi, \psi)\)-contractive condition for a mappings having the mixed \(g\)-monotone property. We illustrate our results with the help of an example.

Keywords: Tripled coincidence point, partially ordered metric space, mixed \(g\)-monotone property

1 Introduction

The Banach contraction principle is the most celebrated fixed point theorem. Afterward many authors obtained many important extensions of this principle (cf. [1]-[15]). Recently Bhaskar and Lakshmikantham [4], Nieto and Lopez [11]-[12], Ran and Reurings [13] and Agarwal, El-Gebeily and O’Regan [2] presented some new results for contractions in partially ordered metric spaces. Bhaskar and Lakshmikantham [4] noted that their theorem can be used to investigate a large class of problems and have discussed the existence and uniqueness of solution for a periodic boundary value problem.

Recently, Luong and Thuan [10] presented some coupled fixed point theorems for a mixed monotone mapping in a partially ordered metric space which are generalizations of the results of Bhaskar and Lakshmikantham [4]. In this paper, we establish the existence and uniqueness of coupled coincidence point involving a \((\phi, \psi)\)-contractive condition for mappings having the mixed \(g\)-monotone property. We also illustrate our results with the help of an example.
2 Preliminaries

A partial order is a binary relation \( \preceq \) over a set \( X \) which is reflexive, anti-symmetric, and transitive. Now, let us recall the definition of the monotonic function \( f : X \to X \) in the partially order set \( (X, \preceq) \). We say that \( f \) is non-decreasing if for \( x, y \in X, x \preceq y \), we have \( fx \preceq fy \). Similarly, we say that \( f \) is non-increasing if for \( x, y \in X, x \preceq y \), we have \( fx \succeq fy \). Any one could read on [8] for more details on fixed point theory.

**Definition 2.1** [9] (Mixed g-Monotone Property)
Let \( (X, \preceq) \) be a partially ordered set and \( F : X \times X \times X \to X \). We say that the mapping \( F \) has the mixed g-monotone property if \( F \) is monotone g-non-decreasing in its first argument, \( F \) is monotone g-non-increasing in its second argument and \( F \) is monotone g-non-decreasing in its third argument. That is, for any \( x, y \in X \),

\[
x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z) \tag{1}
\]

\[
y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow F(x, y_1, z) \preceq F(x, y_2, z), \tag{2}
\]

and

\[
z_1, z_2 \in X, gz_1 \preceq gz_2 \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2) \tag{3}
\]

**Definition 2.2** [9] (Tripled Coincidence Point)
Let \( (X, \preceq) \) be a partially ordered set and \( F : X \times X \times X \to X \) and \( g : X \to X \). We say that \( (x, y, z) \) is a tripled coincidence point of \( F \) and \( g \) if \( F(x, y, z) = gx \), \( F(y, x, z) = gy \) and \( F(z, y, x) = gz \) for \( x, y, z \in X \).

**Definition 2.3** [9] Let \( X \) be a non-empty set and let \( F : X \times X \times X \to X \) and \( g : X \to X \). We say \( F \) and \( g \) are commutative if, for all \( x, y, z \in X \),

\[
g(F(x, y, z)) = F(gx, gy, gz).
\]

**Definition 2.4** [5] The mapping \( F \) and \( g \) where \( F : X \times X \times X \to X \) and \( g : X \to X \), are said to be compatible if

\[
\lim_{n \to \infty} d(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n)) = 0,
\]

\[
\lim_{n \to \infty} d(g(F(y_n, x_n, z_n)), F(gy_n, gx_n, gz_n)) = 0,
\]

and

\[
\lim_{n \to \infty} d(g(F(z_n, y_n, x_n)), F(gz_n, gy_n, gx_n)) = 0
\]

whenever \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) are sequences in \( X \), such that \( \lim_{n \to \infty} F(x_n, y_n, z_n) = \lim_{n \to \infty} gx_n = x \), \( \lim_{n \to \infty} F(y_n, x_n, z_n) = \lim_{n \to \infty} gy_n = y \), and \( \lim_{n \to \infty} F(z_n, y_n, x_n) = \lim_{n \to \infty} gz_n = z \) for all \( x, y, z \in X \) are satisfied.
3 Existence of Tripled Coincidence Points

As in [10], let \( \phi \) denote all functions \( \phi : [0, \infty) \to [0, \infty) \) which satisfy

(i) \( \phi \) is continuous and non-decreasing,

(ii) \( \phi(t) = 0 \) if and only if \( t = 0 \),

(iii) \( \phi(t + s) \leq \phi(t) + \phi(s), \forall t, s \in [0, \infty) \),

and let \( \psi \) denote all functions \( \psi : [0, \infty) \to (0, \infty) \) which satisfy \( \lim_{t \to r} \psi(t) > 0 \) for all \( r > 0 \) and \( \lim_{t \to 0^+} \psi(t) = 0 \).

For example [10], functions \( \phi_1(t) = kt \) where \( k > 0 \), \( \phi_2(t) = \frac{1}{t+1} \), \( \phi_3(t) = \ln(t+1) \), and \( \phi_4(t) = \min\{t, 1\} \) are in \( \Phi \); \( \psi_1(t) = kt \) where \( k > 0 \), \( \psi_2(t) = \frac{\ln(2t+1)}{2} \), and

\[
\psi_3(t) = \begin{cases} 
1, & t = 0 \\
\frac{t}{t+1}, & 0 < t < 1 \\
1, & t = 1 \\
\frac{1}{2}t, & t > 1 
\end{cases}
\]

are in \( \Psi \).

Now let us start proving our main results.

**Theorem 3.1** Let \( (X, \leq) \) be a partially ordered set and suppose there is a metric \( d \) on \( X \) such that \( (X, d) \) is a complete metric space. Let \( F : X \times X \times X \to X \) be a mapping having the mixed \( g \)-monotone property on \( X \) such that there exist three elements \( x_0, y_0, z_0 \in X \) with \( gx_0 \leq F(x_0, y_0, z_0), gy_0 \geq F(y_0, x_0, z_0) \) and \( gz_0 \leq F(z_0, y_0, x_0) \).

Suppose there exist \( \phi \in \Phi \) and \( \psi \in \Psi \) such that

\[
\phi(d(F(x, y, z), F(u, v, w))) \leq \frac{1}{3} \phi(d(gx, gu) + d(gy, gv) + d(gz, gw))
- \psi \left( \frac{d(gx, gu) + d(gy, gv) + d(gz, gw)}{3} \right)
\]

(4)

for all \( x, y, z, u, v, w \in X \) with \( gx \geq gu, gy \leq gv \) and \( gz \geq gw \). Suppose \( F(X \times X \times X) \subseteq g(X) \), \( g \) is continuous and compatible with \( F \) and also suppose either

(a) \( F \) is continuous

(b) \( X \) has the following property:

(i) if a non-decreasing sequence \( \{x_n\} \to x \), then \( x_n \leq x \), for all \( n \),

(ii) if a non-increasing sequence \( \{y_n\} \to y \), then \( y \leq y_n \), for all \( n \),

(iii) if a non-decreasing sequence \( \{z_n\} \to z \), then \( z_n \leq z \), for all \( n \),

then there exist \( x, y, z \in X \) such that

\[
gx = F(x, y, z) \quad gy = F(y, x, z) \quad \text{and} \quad gz = F(z, y, x)
\]

that is, \( F \) and \( g \) have a tripled coincidence point in \( X \).
Proof. Let \( x_0, y_0, z_0 \in X \) be such that \( gx_0 \preceq F(x_0, y_0, z_0) \) and \( gy_0 \succeq F(y_0, x_0, z_0) \) and \( gz_0 \preceq F(z_0, y_0, x_0) \). Using \( F(X \times X \times X) \subseteq g(X) \), we construct sequences \( \{x_n\}, \{y_n\} \) and \( \{z_n\} \) in \( X \) as, for all \( n \geq 0 \),

\[
gx_{n+1} = F(x_n, y_n, z_n), \quad gy_{n+1} = F(y_n, x_n, z_n) \quad \text{and} \quad g\z_{n+1} = F(z_n, y_n, x_n). \quad (5)
\]

We are going to prove that, for all \( n \geq 0 \),

\[
gx_n \preceq gx_{n+1}, \quad (6)
\]

\[
 gy_n \succeq gy_{n+1} \quad (7)
\]

and

\[
 g\z_n \preceq g\z_{n+1}. \quad (8)
\]

To prove these, we are going to use the mathematical induction.

Let \( n = 0 \). Since \( gx_0 \preceq F(x_0, y_0, z_0) \), \( gy_0 \succeq F(y_0, x_0, z_0) \) and \( g\z_0 \preceq F(z_0, y_0, x_0) \) and since \( gx_1 = F(x_0, y_0, z_0) \), \( gy_1 = F(y_0, x_0, z_0) \) and \( g\z_1 = F(z_0, y_0, x_0) \), we have \( gx_0 \preceq gx_1 \), \( gy_0 \succeq gy_1 \) and \( g\z_0 \preceq g\z_1 \). Thus (6), (7) and (8) hold for \( n = 0 \).

Suppose now that (6), (7) and (8) hold for some fixed \( n \geq 0 \). Then, since \( gx_n \preceq gx_{n+1}, gy_n \succeq gy_{n+1} \) and \( g\z_n \preceq g\z_{n+1} \) and by mixed \( g\)-monotone property of \( F \), we have

\[
 gx_{n+2} = F(x_{n+1}, y_{n+1}, z_{n+1}) \succeq F(x_n, y_{n+1}, z_{n+1}) \succeq F(x_n, y_n, z_n) = gx_{n+1}, \quad (9)
\]

\[
 gy_{n+2} = F(y_{n+1}, x_{n+1}, z_{n+1}) \preceq F(y_n, x_{n+1}, z_{n+1}) \preceq F(y_n, x_n, z_n) = gy_{n+1} \quad (10)
\]

and

\[
 g\z_{n+2} = F(z_{n+1}, y_{n+1}, x_{n+1}) \succeq F(z_n, y_{n+1}, x_{n+1}) \succeq F(z_n, y_n, x_n) = g\z_{n+1}. \quad (11)
\]

Using (9), (10) and (11), we get

\[
 gx_{n+1} \preceq gx_{n+2}, \quad gy_{n+1} \succeq gy_{n+2} \quad \text{and} \quad g\z_{n+1} \preceq g\z_{n+2}.
\]

Hence by the mathematical induction we conclude that (6), (7) and (8) hold for all \( n \geq 0 \). Therefore,

\[
 gx_0 \preceq gx_1 \preceq gx_2 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots, \quad (12)
\]

\[
 gy_0 \succeq gy_1 \succeq gy_2 \succeq \cdots \succeq gy_n \succeq gy_{n+1} \succeq \cdots \quad (13)
\]

and

\[
 g\z_0 \preceq g\z_1 \preceq g\z_2 \preceq \cdots \preceq g\z_n \preceq g\z_{n+1} \preceq \cdots. \quad (14)
\]
Since $gx_n \succeq gx_{n-1}$, $gy_n \preceq gy_{n-1}$, $gz_n \succeq gz_{n-1}$ and using (4) and (5), we have

$$
\phi(d(gx_{n+1}, gx_n)) = \phi(d(F(x_n, y_n, z_n), F(x_{n-1}, y_{n-1}, z_{n-1}))) \\
\leq \frac{1}{3} \phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})) \\
- \psi \left( \frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})}{3} \right).
$$

(15)

Similarly, since $gy_{n-1} \succeq gy_n$, $gx_{n-1} \preceq gx_n$, $gz_n \succeq gz_{n-1}$ and using (4) and (5), we also have

$$
\phi(d(gy_n, gy_{n+1})) = \phi(d(F(y_{n-1}, x_{n-1}, z_{n-1}), F(y_n, x_n, z_n))) \\
\leq \frac{1}{3} \phi(d(gy_{n-1}, gy_n) + d(gx_{n-1}, gx_n) + d(gz_n, gz_{n-1})) \\
- \psi \left( \frac{d(gy_{n-1}, gy_n) + d(gx_{n-1}, gx_n) + d(gz_n, gz_{n-1})}{3} \right).
$$

(16)

Similarly, since $gz_n \succeq gz_{n-1}, gy_{n-1} \succeq gy_n$, $gx_{n-1} \preceq gx_n$, and using (4) and (5), we also have

$$
\phi(d(gz_{n+1}, gz_n)) = \phi(d(F(z_n, y_n, x_n), F(z_{n-1}, y_{n-1}, x_{n-1}))) \\
\leq \frac{1}{3} \phi(d(gz_n, gz_{n-1}) + d(gy_n, gy_{n-1}) + d(gx_n, gx_{n-1})) \\
- \psi \left( \frac{d(gz_n, gz_{n-1}) + d(gy_n, gy_{n-1}) + d(gx_n, gx_{n-1})}{3} \right).
$$

(17)

Using (15), (16) and (17), we have

$$
\phi(d(gx_{n+1}, gx_n)) + \phi(d(gy_{n+1}, gy_n)) + \phi(d(gz_{n+1}, gz_n)) \\
\leq \phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})) \\
- 3\psi \left( \frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})}{3} \right).
$$

(18)

By property (iii) of $\phi$, we have

$$
\phi(d(gx_{n+1}, gx_n)) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)) \\
\leq \phi(d(gx_{n+1}, gx_n)) + \phi(d(gy_{n+1}, gy_n)) + \phi(d(gz_{n+1}, gz_n)).
$$

(19)
Using (18) and (19), we have
\[
\begin{align*}
\phi(d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)) &
\leq \phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})) \\
&- 3\psi \left( \frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})}{3} \right).
\end{align*}
\]  

(20)

which implies, since \( \psi \) is a non-negative function,
\[
\begin{align*}
\phi(d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)) &
\leq \phi(d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1})).
\end{align*}
\]

Using the fact that \( \phi \) is non-decreasing, we get
\[
\begin{align*}
d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n) &
\leq d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1}).
\end{align*}
\]

Set
\[
\delta_n = d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n).
\]

Now we would like to show that \( \delta_n \to 0 \) as \( n \to \infty \). It is clear that the sequence \( \{ \delta_n \} \) is decreasing. Therefore, there is some \( \delta \geq 0 \) such that
\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} [d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)] = \delta. \quad (21)
\]

We shall show that \( \delta = 0 \). Suppose, to the contrary, that \( \delta > 0 \). Then taking the limit as \( n \to \infty \) (equivalently, \( \delta_n \to \delta \)) of both sides of (20) and remembering \( \lim_{t \to r} \psi(t) > 0 \) for all \( r > 0 \) and \( \phi \) is continuous, we have
\[
\begin{align*}
\phi(\delta) &= \lim_{n \to \infty} \phi(\delta_n) \\
&\leq \lim_{n \to \infty} \left[ \phi(\delta_{n-1}) - 3\psi \left( \frac{\delta_{n-1}}{3} \right) \right] \\
&= \phi(\delta) - 3 \lim_{\delta_{n-1} \to \delta} \psi \left( \frac{\delta_{n-1}}{3} \right) < \phi(\delta)
\end{align*}
\]

a contradiction. Thus \( \delta = 0 \), that is
\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} [d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n) + d(gz_{n+1}, gz_n)] = 0. \quad (22)
\]

Now, we will prove that \( \{ gx_n \} \), \( \{ gy_n \} \) and \( \{ gz_n \} \) are Cauchy sequences. Suppose, to the contrary, that at least one of \( \{ gx_n \} \), \( \{ gy_n \} \) or \( \{ gz_n \} \) is not a Cauchy sequence. Then there exists an \( \epsilon > 0 \) for which we can find subsequences \( \{ gx_{n(k)} \} \), \( \{ gx_{m(k)} \} \) of \( \{ gx_n \} \), \( \{ gy_{n(k)} \} \), \( \{ gy_{m(k)} \} \) of \( \{ gy_n \} \) and \( \{ gz_{n(k)} \} \), \( \{ gz_{m(k)} \} \) of \( \{ gz_n \} \) with \( n(k) > m(k) \geq k \) such that
\[
d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gz_{m(k)}) \geq \epsilon. \quad (23)
\]
Further, corresponding to \( m(k) \), we can choose \( n(k) \) in such a way that it is the smallest integer with \( n(k) > m(k) \) and satisfying (23). Then

\[
d(gx_{n(k)-1}, gx_{m(k)}) + d(gy_{n(k)-1}, gy_{m(k)}) + d(gz_{n(k)-1}, gz_{m(k)}) < \epsilon. \tag{24}
\]

Using (23), (24) and the triangle inequality, we have

\[
\epsilon \leq r_k := d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gz_{m(k)})
\leq d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)}) + d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)})
+ d(gz_{n(k)}, gzm_{n(k)-1}) + d(gzm_{n(k)-1}, gzm_{m(k)})
\leq d(gx_{n(k)}, gx_{n(k)-1}) + d(gy_{n(k)}, gy_{n(k)-1}) + d(gz_{n(k)}, gzm_{n(k)-1}) + \epsilon.
\]

Letting \( k \to \infty \) and using (22), we get

\[
\lim_{k \to \infty} r_k = \lim_{k \to \infty} [d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gzm_{m(k))}] = \epsilon. \tag{25}
\]

By the triangle inequality

\[
r_k = d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gzm_{m(k)})
\leq d(gx_{n(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)})
+ d(gy_{n(k)}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{m(k)})
+ d(gz_{n(k)}, gzm_{n(k)+1}) + d(gzm_{n(k)+1}, gzm_{m(k)+1}) + d(gzm_{m(k)+1}, gzm_{m(k)})
= \delta_{n(k)} + \delta_{m(k)} + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1})
+ d(gz_{n(k)+1}, gzm_{m(k)+1}).
\]

Using the property of \( \phi \), we have

\[
\phi(r_k) = \phi(\delta_{n(k)} + \delta_{m(k)} + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1})
+ d(gz_{n(k)+1}, gzm_{m(k)+1}))
\leq \phi(\delta_{n(k)} + \delta_{m(k)}) + \phi(d(gx_{n(k)+1}, gx_{m(k)+1}))
+ \phi(d(gy_{n(k)+1}, gy_{m(k)+1})) + \phi(d(gz_{n(k)+1}, gzm_{m(k)+1})). \tag{26}
\]

Since \( n(k) > m(k) \), hence \( gx_{n(k)} \geq gx_{m(k)} \), \( gy_{n(k)} \leq gy_{m(k)} \) and \( gzm_{n(k)} \geq gzm_{m(k)} \).

Using (4) and (5), we get

\[
\phi(d(gx_{n(k)+1}, gx_{m(k)+1}))
= \phi(d(F(x_{n(k)}, y_{n(k)}, z_{n(k)}), F(x_{m(k)}, y_{m(k)}, z_{n(k)})))
\leq \frac{1}{3} \phi(d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gzm_{m(k)}))
- \psi \left( \frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gz_{n(k)}, gzm_{m(k)})}{3} \right)
= \frac{1}{3} \phi(r_k) - \psi \left( \frac{r_k}{3} \right). \tag{27}
\]
By the same way, we also have
\[
\begin{align*}
\phi(d(gy_{n(k)+1}, gy_{n(k)})) \\
= \phi(d(F(y_{n(k)}, x_{n(k)}), F(y_{n(k)}, x_{n(k)}), z_{n(k)})) \\
\leq \frac{1}{3} \phi(d(gy_{n(k)}, gy_{n(k)}) + d(gx_{n(k)}, gx_{n(k)}) + d(gz_{n(k)}, gz_{n(k)})) \\
- \psi \left( \frac{d(gy_{n(k)}, gy_{n(k)}) + d(gx_{n(k)}, gx_{n(k)}) + d(gz_{n(k)}, gz_{n(k)})}{3} \right) \\
= \frac{1}{3} \phi(r_k) - \psi \left( \frac{r_k}{3} \right).
\end{align*}
\]
(28)

Also by the same way, we also have
\[
\begin{align*}
\phi(d(gz_{n(k)+1}, gz_{n(k)})) \\
= \phi(d(F(z_{n(k)}, y_{n(k)}), F(z_{n(k)}, y_{n(k)}), x_{n(k)})) \\
\leq \frac{1}{3} \phi(d(gz_{n(k)}, gz_{n(k)}) + d(gy_{n(k)}, gy_{n(k)}) + d(gx_{n(k)}, gx_{n(k)})) \\
- \psi \left( \frac{d(gz_{n(k)}, gz_{n(k)}) + d(gy_{n(k)}, gy_{n(k)}) + d(gx_{n(k)}, gx_{n(k)})}{3} \right) \\
= \frac{1}{3} \phi(r_k) - \psi \left( \frac{r_k}{3} \right).
\end{align*}
\]
(29)

Inserting (27), (28) and (29) in (26), we have
\[
\phi(r_k) \leq \phi(\delta_n) + \delta_m + \phi(r_k) - 3\psi \left( \frac{r_k}{3} \right).
\]

Letting \( k \to \infty \) and using (22) and (25), we get
\[
\phi(\epsilon) \leq \phi(0) + \phi(\epsilon) - 3 \lim_{k \to \infty} \psi \left( \frac{r_k}{3} \right) = \phi(\epsilon) - 3 \lim_{r_k \to \epsilon} \psi \left( \frac{r_k}{3} \right) < \phi(\epsilon)
\]
a contradiction. This shows that \( \{gx_n\}, \{gy_n\} \) and \( \{gz_n\} \) are Cauchy sequences. Since \( X \) is a complete metric space, there exist \( x, y, z \in X \) such that
\[
\begin{align*}
\lim_{n \to \infty} F(x_n, y_n, z_n) = \lim_{n \to \infty} gx_n = x, & \quad \lim_{n \to \infty} F(y_n, x_n, z_n) = \lim_{n \to \infty} gy_n = y, \\
\lim_{n \to \infty} F(z_n, y_n, x_n) = \lim_{n \to \infty} gz_n = z.
\end{align*}
\]
(30)

Since \( F \) and \( g \) are compatible mappings, we have
\[
\begin{align*}
\lim_{n \to \infty} d(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n)) = 0, \\
\lim_{n \to \infty} d(g(F(y_n, x_n, z_n)), F(gy_n, gx_n, gz_n)) = 0
\end{align*}
\]
(31)
and
\[ \lim_{n \to \infty} d(g(F(z_n, y_n, x_n)), F(gz_n, gy_n, gx_n)) = 0. \quad (33) \]

We now show that \( gx = F(x, y, z) \), \( gy = F(y, x, z) \) and \( gz = F(z, y, x) \). Suppose that the assumption (a) holds. For all \( n \geq 0 \), we have,
\[
d(gx, F(gx_n, gy_n)) \leq d(gx, g(F(x_n, y_n))) + d(g(F(x_n, y_n)), F(gx_n, gy_n)).
\]
Taking the limit as \( n \to \infty \), using (5), (30), (31) and the fact that \( F \) and \( g \) are continuous, we have
\[ d(gx, F(x, y, z)) = 0. \]
Similarly, using (5), (30), (32) and the fact that \( F \) and \( g \) are continuous, we have
\[ d(gy, F(y, x, z)) = 0. \]
Similarly, using (5), (30), (33) and the fact that \( F \) and \( g \) are continuous, we have
\[ d(gz, F(z, y, x)) = 0. \]
From all of these, we get
\[ gx = F(x, y, z) \quad gy = F(y, x, z) \quad \text{and} \quad gz = F(z, y, x). \]

Finally, suppose that (b) holds. By (6), (7) and (30), we have \( \{gx_n\} \) is a non-decreasing sequence, \( gx_n \to x \), \( \{gy_n\} \) is a non-increasing sequence, \( gy_n \to y \) and \( \{gz_n\} \) is a non-decreasing sequence, \( gz_n \to z \) as \( n \to \infty \). Hence, by assumption (b), we have for all \( n \geq 0 \),
\[ gx_n \preceq x, \quad gy_n \succeq y \quad \text{and} \quad gz_n \preceq z. \quad (34) \]
Since \( F \) and \( g \) are compatible mappings and \( g \) is continuous, by (31), (32) and (33) we have
\[ \lim_{n \to \infty} g(gx_n) = gx = \lim_{n \to \infty} g(F(x_n, y_n, z_n)) = \lim_{n \to \infty} F(gx_n, gy_n, gz_n), \quad (35) \]
\[ \lim_{n \to \infty} g(gy_n) = gy = \lim_{n \to \infty} g(F(y_n, x_n, z_n)) = \lim_{n \to \infty} F(gy_n, gx_n, gz_n). \quad (36) \]
and
\[ \lim_{n \to \infty} g(gz_n) = gz = \lim_{n \to \infty} g(F(z_n, y_n, x_n)) = \lim_{n \to \infty} F(gz_n, gy_n, gx_n). \quad (37) \]
Now we have
\[ d(gx, F(x, y, z)) \leq d(gx, g(gx_{n+1})) + d(g(gx_{n+1}), F(x, y, z)). \]
Taking \( n \to \infty \) in the above inequality, using (5) and (26) we have,
\[
d(gx, F(x, y, z)) \leq \lim_{n \to \infty} d(gx, g(gx_{n+1})) + \lim_{n \to \infty} d(g(F(x_n, y_n, z_n)), F(x, y, z)) \\
\leq \lim_{n \to \infty} d(F(gx_n, gy_n, gz_n)), F(x, y, z)). \quad (38) \]
Using the property of \( \phi \), we get
\[ \phi(d(gx, F(x, y, z))) \leq \lim_{n \to \infty} \phi(d(F(gx_n, gy_n, gz_n)), F(x, y, z))). \]
Since the mapping $g$ is monotone increasing, using (4), (34) and (38), we have for all $n \geq 0$,
\[
\phi(d(gx, F(x, y, z))) \leq \lim_{n \to \infty} \frac{1}{3} \phi(d(ggx_n, gx) + d(ggy_n, gy) + d(ggz_n, gz))
- \lim_{n \to \infty} \psi \left( \frac{d(ggx_n, gx) + d(ggy_n, gy) + d(ggz_n, gz)}{3} \right).
\]

Using the above inequality, (30) and the property of $\phi$, we get $\phi(d(gx, F(x, y, z))) = 0$, thus $d(gx, F(x, y, z)) = 0$. Hence $gx = F(x, y, z)$.

Similarly, we can show that $gy = F(y, x, z)$ and $gz = F(z, y, x)$. Thus we proved that $F$ and $g$ have a tripled coincidence point.

4 Uniqueness of Tripled Coincidence Point

In this section, we will prove the uniqueness of the tripled coincidence point. Note that if $(X, \preceq)$ is a partially ordered set, then we endow the product $X \times X \times X$ with the following partial order relation, for all $(x, y, z), (u, v, w) \in X \times X \times X$,

\[(x, y, z) \preceq (u, v, w) \iff x \preceq u, y \preceq v, z \preceq w.\]

**Theorem 4.1** In addition to hypotheses of Theorem 3.1, suppose that for every $(x, y, z), (x_1, y_1, z_1)$ in $X \times X \times X$ there exists a $(u, v, w)$ in $X \times X \times X$ that is comparable to $(x, y, z)$ and $(x_1, y_1, z_1)$, then $F$ and $g$ have a unique tripled coincidence point.

**Proof.** From Theorem 3.1, the set of tripled coincidence points of $F$ and $g$ is non-empty. Suppose $(x, y, z)$ and $(x_1, y_1, z_1)$ are tripled coincidence points of $F$ and $g$, that is $gx = F(x, y, z), gy = F(y, x, z), gz = F(z, y, x), gx_1 = F(x_1, y_1, z_1), gy_1 = F(y_1, x_1, z_1)$ and $gz_1 = F(z_1, y_1, x_1)$. We are going to show that $gx = gx_1, gy = gy_1$ and $gz = gz_1$. By assumption, there exists $(u, v, w) \in X \times X \times X$ that is comparable to $(x, y, z)$ and $(x_1, y_1, z_1)$. We define sequences \(\{gu_n\}, \{gv_n\}\) and \(\{gw_n\}\) as follows

\[
u_0 = u, \quad v_0 = v, \quad w_0 = w, \quad gu_{n+1} = F(u_n, v_n, w_n), \quad gv_{n+1} = F(v_n, u_n, w_n)
\]

and \(gw_{n+1} = F(w_n, v_n, u_n)\) for all $n$.

Since $(u, v, w)$ is comparable with $(x, y, z)$, we may assume that $(x, y, z) \succeq (u, v, w) = (u_0, v_0, w_0)$. Using the mathematical induction, it is easy to prove that

\[(x, y, z) \succeq (u_n, v_n, w_n) \text{ for all } n. \tag{39}\]
Using (4) and (39), we have

\[
\varphi(d(gx, gu_{n+1})) = \varphi(d(F(x, y, z), F(u_n, v_n, w_n)))
\leq \frac{1}{3} \varphi(d(x, u_n) + d(y, v_n) + d(z, w_n)) \\
- \psi \left( \frac{d(x, u_n) + d(y, v_n) + d(z, w_n)}{3} \right).
\] (40)

Similarly

\[
\varphi(d(gv_{n+1}, gy)) = \varphi(d(F(v_n, u_n, w_n), F(y, x, z)))
\leq \frac{1}{3} \varphi(d(v_n, y) + d(u_n, x) + d(w_n, z)) \\
- \psi \left( \frac{d(v_n, y) + d(u_n, x) + d(w_n, z)}{3} \right).
\] (41)

\[
\varphi(d(gz, gw_{n+1})) = \varphi(d(F(z, y, x), F(w_n, v_n, u_n)))
\leq \frac{1}{3} \varphi(d(z, w_n) + d(y, v_n) + d(x, u_n)) \\
- \psi \left( \frac{d(z, w_n) + d(y, v_n) + d(x, u_n)}{3} \right).
\] (42)

Using (40), (41), (42) and the property of \( \varphi \), we have

\[
\varphi(d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})) \\
\leq \varphi(d(gx, gu_{n+1}) + \varphi(d(gy, gv_{n+1})) + \varphi(d(gz, gw_{n+1})) \\
\leq \varphi(d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)) \\
- 3\psi \left( \frac{d(gx, gu_n) + d(gy, gv_n)}{3} \right).
\] (43)

which implies, using the property of \( \psi \),

\[
\varphi(d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1})) \leq \varphi(d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)).
\]

Thus, using the property of \( \phi \),

\[
d(gx, gu_{n+1}) + d(gy, gv_{n+1}) + d(gz, gw_{n+1}) \leq d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n).
\]

That is the sequence \( \{d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)\} \) is decreasing. Therefore, there exists \( \alpha \geq 0 \) such that

\[
\lim_{n \to \infty} [d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)] = \alpha.
\] (44)
We will show that \( \alpha = 0 \). Suppose, to the contrary, that \( \alpha > 0 \). Taking the limit as \( n \to \infty \) in (43), we have, using the property of \( \psi \),

\[
\varphi(\alpha) \leq \varphi(\alpha) - 3 \lim_{n \to \infty} \psi \left( \frac{d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n)}{3} \right) < \varphi(\alpha)
\]

a contradiction. Thus, \( \alpha = 0 \), that is,

\[
\lim_{n \to \infty} \left[ d(gx, gu_n) + d(gy, gv_n) + d(gz, gw_n) \right] = 0.
\]

It implies

\[
\lim_{n \to \infty} d(gx, gu_n) = \lim_{n \to \infty} d(gy, gv_n) = \lim_{n \to \infty} d(gz, gw_n) = 0.
\]  \hspace{1cm} (45)

Similarly, we show that

\[
\lim_{n \to \infty} d(gx_1, gu_n) = \lim_{n \to \infty} d(gy_1, gv_n) = \lim_{n \to \infty} d(gz_1, gw_n) = 0.
\]  \hspace{1cm} (46)

Using (45) and (46) we have \( gx = gx_1 \), \( gy = gy_1 \) and \( gz = gz_1 \).

**Corollary 4.1** [10] In addition to hypotheses of Theorem 3.1, suppose that for every \((x, y, z), (x_1, y_1, z_1)\) in \( X \times X \times X \), there exists a \((u, v, w)\) in \( X \times X \times X \) that is comparable to \((x, y, z)\) and \((x_1, y_1, z_1)\), then \( F \) and \( g \) have a unique tripled coincidence point.

5 **Example**

**Example 5.1** Let \( X = \mathbb{R} \). Then \( (X, \leq) \) is a partially ordered set with the natural ordering of real numbers. Let

\[
d(x, y) = |x - y| \quad \text{for} \quad x, y \in X.
\]

Then \((X, d)\) is a complete metric space.

Let \( g : X \to X \) be defined as

\[
gx = x^2, \text{ for all } x \in X,
\]

and let \( F : X \times X \times X \to X \) be defined as

\[
F(x, y, z) = \frac{2x^2 - 2y^2 + 8z^2 + 1}{3}
\]

\( F \) obeys the mixed \( g \)-monotone property.

It is easy to check that \((\frac{1}{4}, \frac{1}{4}, \frac{1}{4})\) is the unique tripled coincidence point of \( F \) and \( g \).
References


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