On the k-Bessel Functions

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Abstract
In this brief paper introduces some k-generalizations of the so-called
special functions as Bessel functions and the Fox-Wright functions.

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I Introduction and Preliminaries

Since Diaz and Pariguan (cf.[2]) have introduced the k-gamma function
Γ_k(z) and the generalized Pochhammer k-symbol, several articles have been
devoted to studying generalizations of some of the so-called special functions.
So can be found the k-Beta function, the k-Zeta function, the k-Mittag-Leffler
function and the k-Wright function.

The integral expression of the k-gamma function is given by

\[ \Gamma_k(z) = \int_0^\infty e^{-\frac{z}{k}} t^{z-1} dt, \quad Re(z) > 0, \ k > 0. \] (I.1)

Whose relationship with the classical Gamma Euler functions is given by

\[ \Gamma_k(z) = k^{\frac{z}{k}} \Gamma\left(\frac{z}{k}\right) \] (I.2)

We collect some of its properties in the following

Lemma 1 The k-gamma function \( \Gamma_k(z) \) verified that:
1. \( \Gamma_k(z + k) = z\Gamma(z) \)
2. \( \Gamma_k(k) = 1 \)

3. Let \( a \in \mathbb{R} \),
   \[
   \Gamma_k(z) = a^z \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} \, dt. \tag{I.3}
   \]

4. \( \Gamma_k(z) \Gamma_k(k-z) = \frac{\pi}{\sin(\pi z/k)} \)

For the proof, that we omit, we refer to [2].

Right now we also have the k-Beta function \( B_k(z) \) that is defined by the formula
   \[
   B_k(z, w) = \frac{\Gamma_k(z) \Gamma_k(w)}{\Gamma_k(z + w)}, \quad \text{Re}(z) > 0, \, \text{Re}(w) > 0. \tag{I.4}
   \]

that have the integral representation given by
   \[
   B_k(z, w) = \frac{1}{k} \int_0^\infty t^{z-1} (1 - t)^{w-1} \, dt \tag{I.5}
   \]

Two functions widely used in fractional calculus because of the importance of their roles in the solution of fractional differential equations are the Mittag-Leffler function \( E_\alpha(z) \) and the Wright functions \( W(z) \).

The Mittag-Leffler function is an entire function defined by
   \[
   E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \tag{I.6}
   \]

A first generalization is given by a more general series
   \[
   E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \, \beta > 0. \tag{I.7}
   \]

called the Mittag-Leffler of two parameters.

From (I.6) and (I.7) we have
   \[
   E_{\alpha,1}(z) = E_\alpha(z)
   \]

Another generalization was done by Prabhakar (cf.[7]) who introduced the Mittag-Leffler type function \( E^\gamma_\alpha(z) \) defined by
   \[
   E^\gamma_\alpha(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta) n!} \frac{z^n}{n!}, \quad \text{Re}(\alpha) > 0, \, \text{Re}(\beta) > 0 \tag{I.8}
   \]
with \((\gamma)_n\) the Pochhammer symbol given by

\[
(\gamma)_n = \gamma(\gamma + 1)(\gamma + 2)\cdots(\gamma + n - 1) = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} \quad (I.9)
\]

In a recent paper of us (cf.[3]) we have defined a new Mittag-Leffler type function as the series

\[
E^\gamma_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{n!} \quad (I.10)
\]

valid for \(Re(\alpha) > 0, Re(\beta) > 0, \gamma \in \mathbb{C}\) and \((\gamma)_{n,k}\) is the Pochhammer k-symbol given by

\[
(\gamma)_{n,k} = \gamma(\gamma + k)(\gamma + 2k)\cdots(\gamma + (n - 1)k), \quad \gamma \in \mathbb{C}, \; k \in \mathbb{R}, \; n \in \mathbb{N}. \quad (I.11)
\]

It may be observed that \(E^\gamma_{k,\alpha,\beta}(z)\) is such that \(E^\gamma_{k,\alpha,\beta}(z) \rightarrow E^\gamma_{\alpha,\beta}(z)\) as \(k \rightarrow 1\), since \((\gamma)_{n,k} \rightarrow (\gamma)_n\), \(\Gamma_k(z) \rightarrow \Gamma(z)\) and the convergence of the series in (I.10) is uniform on compact subsets.

Also we have defined (cf.[?]) the k-Wright type function as the series

\[
W^\gamma_{k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma_k(\alpha n + \beta)} \frac{z^n}{(n!)^2} \quad (I.12)
\]

for \(Re(\alpha) > -1, \; Re(\beta) > 0, \; k \in \mathbb{R}, \; n \in \mathbb{N}\).

Can be easily seen that when \(\gamma = 1\) and \(k = 1\) (I.12) reduces to the classical Wright function

\[
W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \quad (I.13)
\]

II k-Bessel functions

Based on the well know relation (cf.[5])

\[
J_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} W_{1,\nu+1}\left(-\frac{z^2}{4}\right) \quad (II.1)
\]

where \(W_{\lambda,\nu}(z)\) is the Wright function defined in (I.13) and \(J_{\nu}(z)\) is the Bessel function of the first kind of order \(\nu\) (cf.[?]) given by the series
we have defined (cf. [1]) the k-Bessel function of the first kind \( J_{k,\nu}^{\gamma,\lambda}(z) \) as

\[
J_{k,\nu}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \nu + 1)} \frac{(-1)^n (z/2)^n}{(n!)^2} (n+\nu+1)^{-1} \quad (II.3)
\]

where \((\gamma)_{n,k}\) is the Pochhammer k-symbol and \(\Gamma_k(z)\) is the k-gamma function.

From (I.13) and (II.3) it may be write

\[
J_{k,\nu}^{\gamma,\lambda}(z) = \left(\frac{z}{2}\right)^\nu W_{k,\lambda,\nu+1}^{\gamma} \left(-\frac{z^2}{4}\right). \quad (II.4)
\]

Next, we put the following

**Definition 1** The k-modified Bessel function of the first kind of order \(\nu\) (or \(\nu\) respectively) as

\[
I_{k,\nu}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \nu + k)} \frac{(z/2)^{\nu+2n}}{(n!)^2} (n+\nu+k)^{-1} \quad (II.5)
\]

or

\[
I_{k,-\nu}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n - \nu + k)} \frac{(z/2)^{2n-\nu}}{(n!)^2} (n+\nu+k)^{-1} \quad (II.6)
\]

In terms of the k-Wright function we have

\[
I_{k,\nu}^{\gamma,\lambda}(z) = \pi^{\nu} \left[I_{k,-\nu}^{\gamma,\lambda}(z) - I_{k,\nu}^{\gamma,\lambda}(z) \right] \quad (II.7)
\]

Also we have following

**Definition 2** The k-modified Bessel function of the third kind \(K_{k,\nu}^{\gamma,\lambda}(z)\) is

\[
K_{k,\nu}^{\gamma,\lambda}(z) = \pi \left[I_{k,-\nu}^{\gamma,\lambda}(z) - I_{k,\nu}^{\gamma,\lambda}(z) \right] \quad (II.8)
\]
Now, we will show some elementary properties.

**Lemma 2** Let \( \nu \) be a complex number, \( \text{Re}(\nu) > 0 \) and let \( k, \gamma, z \) be real nonnegative numbers. For \( \lambda = 1 \) holds

\[
\frac{d}{dz} \left( z^{\nu/2} I_{k,\nu}^{1}(\sqrt{z}) \right) = 2^{-k} z^{\frac{\nu-k}{2}} z^{k-1} I_{k,\nu-k}^{1}(\sqrt{z}) \quad (\text{II.9})
\]

*Proof.* From Definition (II.5) we have

\[
z^{\nu/2} I_{k,\nu}^{1}(\sqrt{z}) = z^{\nu/2} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}(\sqrt{z}/2)^{\nu + 2n}}{(n!)^2 \Gamma_k(n + \nu + k)} = \frac{1}{2^\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{(n!)^2 4^n \Gamma_k(n + \nu + k)} z^{\nu + n}. \quad (\text{II.10})
\]

Then

\[
\frac{d}{dz} \left( z^{\nu/2} I_{k,\nu}^{1}(\sqrt{z}) \right) = \frac{1}{2^\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}(\nu + n) z^{\nu + n - 1}}{(n!)^2 \Gamma_k(n + \nu + k)} = 2^{-k} z^{\frac{\nu-k}{2}} z^{k-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}(\sqrt{z}/2)^{(\nu-k)+2n}}{(n!)^2 \Gamma_k(\nu + n)} = 2^{-k} z^{\frac{\nu-k}{2}} z^{k-1} I_{k,\nu-k}^{1}(\sqrt{z}).
\]

**Lemma 3** Let \( I_{k,-\nu}^{1}(z) \) be the \( k \)-modified Bessel function of the first kind of order \(-\nu\), and let \( z \) be a real nonnegative number. Then holds

\[
\frac{d}{dz} \left( z^{\nu/2} I_{k,-\nu}^{1}(\sqrt{z}) \right) = 2^{-k} z^{\frac{\nu-k}{2}} z^{k-1} I_{k,-\nu-k}^{1}(\sqrt{z}) \quad (\text{II.11})
\]

The proof is completely analogous to the Lemma 2 and then we omit it.

**Lemma 4** Let \( \nu \) be a complex number, \( \text{Re}(\nu) > 0 \) and let \( k, \gamma, z \) be real nonnegative numbers, and \( \lambda = 1 \). Then:

\[
\frac{d}{dz} \left[ z^{\nu/2} K_{k,\nu}^{1}(\sqrt{z}) \right] = 2^{-k} z^{\frac{\nu-k}{2}} z^{k-1} K_{k,\nu-k}^{1}(\sqrt{z}) \quad (\text{II.12})
\]
Proof. From (II.8), (II.9) and (II.10) we have
\[ z^{\nu/2} K_{k,\nu}^{\gamma,1}(\sqrt{z}) = z^{\nu/2} \left[ I_{k,-\nu}^{\gamma,1}(\sqrt{z}) - I_{k,\nu}^{\gamma,1}(\sqrt{z}) \right] \frac{\pi}{2 \sin(\nu \pi)} \]

Thus
\[ \frac{d}{dz} \left[ z^{\nu/2} K_{k,\nu}^{\gamma,1}(\sqrt{z}) \right] = \frac{2^{-k} z^{2-\frac{k}{2}} z^{k-1} \pi}{2 \sin(\nu \pi)} \left[ I_{k,-\nu}^{\gamma,1}(\sqrt{z}) - I_{k,\nu}^{\gamma,1}(\sqrt{z}) \right] \]
\[ = \frac{2^{-k} z^{\nu-2} z^{k-1} 2 \sin(\nu + 1) \pi}{2 \sin(\nu \pi)} \left[ I_{k,-\nu}^{\gamma,1}(\sqrt{z}) - I_{k,\nu}^{\gamma,1}(\sqrt{z}) \right] \]
\[ = -2^{-k} z^{\nu-2} z^{k-1} K_{k,\nu}^{\gamma,1}(\sqrt{z}). \]

In the next we will use a fractional integral called k-fractional integral (cf.[6]) that is a k-generalization of the classical Riemann-Liouville fractional integral (cf.[4]).

The k-fractional integral of order \( \alpha \) is defined by
\[ I_{k}^{\alpha}(f)(x) = \frac{1}{k \Gamma_{k}(\alpha)} \int_{0}^{x} (x-t)^{\frac{\alpha}{k}-1} f(t)dt \] (II.13)

It may be observed that when \( k \to 1 \), (II.13) reduces to the classical Riemann-Liouville fractional integral.

Taking into account that (cf.[6])
\[ I_{k}^{\alpha} \left( \frac{x^{\frac{\alpha}{k}}}{\Gamma_{k}(\beta)} \right) = \frac{x^{\frac{\alpha+\beta}{k}}}{\Gamma_{k}(\alpha + \beta)} \] (II.14)

we have the following

**Lemma 5**

\[ I_{k}^{\alpha} \left( z^{\nu/2} I_{k,\nu}^{\gamma,1}(2\sqrt{z}) \right) = z^{\nu+\alpha} k_{1} \Psi_{2} \left[ (k(\nu + 1), k) \right. \right. \left. \left. (\alpha + k(\nu + 1), k), ((\nu + 1), k) \mid z \right] \] (II.15)

**Proof.** By the uniform convergence on compact subsets of the series \( I_{k,\nu}^{\gamma,1}(z) \), from (II.10) and (II.14) we have

\[ I_{k}^{\alpha} \left( z^{\nu/2} I_{k,\nu}^{\gamma,1}(2\sqrt{z}) \right) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{(n!)^{2} \Gamma_{k}(k(\nu + 1) + \alpha + kn) \Gamma_{k}(n + \nu + k)} z^{\frac{\alpha+\nu+n}{k}} \]
\[ = z^{\nu+\alpha} k_{1} \Psi_{2} \left[ (k(\nu + 1), k) \right. \right. \left. \left. (\alpha + k(\nu + 1), k), ((\nu + 1), k) \mid z \right] \]

where \( k_{1} \Psi_{2} \) denote the k-Fox-Wright function.
Among the k-Bessel functions

References

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