In this paper, we give the existence and the representation of the group inverse for circulant block matrix $M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ $(A, B \in K^{n \times n}, \text{and} A^2 = A, B^2 = B)$ over skew field. Some relative additive results are also given.

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1. Introduction

Let $K$ be a skew field and $I$ be the Unit matrix. $K^{n \times n}$ and $A^*$ respectively denote the set of all n-order matrices and the conjugate transpose of $A$ over $K$. For $A \in K^{n \times n}$, the matrix $X \in K^{n \times n}$ is said to be the group inverse of $A$, if $AXA = A, XAX = X, AX = XA$. We then write $X = A^\ast$. It is well known that if $A^\ast$ exists, it is unique, and the conclusion is given in [1].

On representations of the group inverse of block matrices, the authors have made efforts in [2] and [3]. Actually, Generalized inverses have wide applications in many areas such as special matrix theory, singular differential and difference equations and graph theory; see [4], [5], [6], [7] and [8].

Since the problem of finding an explicit representation for the Drazin (group) inverse of a $2 \times 2$ block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (where $A$ and $D$ are required to be square matrices) was proposed by Campbell and Meyer in 1979, considerable progress has been made. A condition for the existence of the group inverse of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is given in [9] under the assumption that $A$ and $I + CA^{-2}B$ are
both invertible over any field; however, the representation of the group inverse is not given. And the representation of the group inverse of the block matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ over skew fields has been given in 2001 in [10]. Though the representation of the Drazin (group) inverse of the block matrix $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ proposed as a problem by Campbell in 1983 in [11] (A is square, 0 is square null matrix) has not been given, there are some achievements about representations of the Drazin (group) inverse of the block matrices $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ under special conditions. Some results are received on matrices over the complex field, e.g. in [12] when $A = B = I_n$ and in [13] when $A, B, C \in P, P^*, PP^*, P^2 = P$. Some results are over skew fields, e.g. in [14] when $A = I_n$ and $\text{rank}(CB)^2 = \text{rank}(B) = \text{rank}(C)$ and in [15] when $A = B, A^2 = A$. In addition, Group inverse of the product of two matrices, as well as some related properties over skew field are given in [16].

In this paper, we mainly give necessary and sufficient conditions for the existence and the representation of the group inverse of a block matrix $M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ ($A, B \in K^{n \times n}$, and $A^2 = A, B^2 = B$), similarly, we also reach a few conclusions under certain conditions.

2. Preliminaries

Lemma 2.1 Suppose $A \in K^{n \times n}$, then $A^i$ exists if and only if $\text{rank}(A) = \text{rank}(A^2)$.

Lemma 2.2 Let $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in K^{n \times n}, A \in k^{r \times r},$ then $M^i$ exists if and only if $A^i, C^i$ exists and $\text{rank}(M) = \text{rank}(A) + \text{rank}(C)$, and then we have $M^i = \begin{pmatrix} A^i & X \\ 0 & C^i \end{pmatrix}$, where $X = (A^i)^2B(I-CC^i) + (I-AA^i)B(CC^i)^2 - A^iBC^i$.

Lemma 2.3 Let $A^2 = A, B^2 = B, and \text{rank}(A-B) \leq \text{rank}[A(I-BA) + B(I-AB)], then (A-B)^i$ exists.

Proof. One part, $\text{rank}[(A-B)^2] \geq \text{rank}[(A-B)^2(A+B)] = \text{rank}[A(I-BA) + B(I-AB)] \geq \text{rank}(A-B); And another part, \text{rank}[(A-B)^2] \leq \text{rank}(A-B)$ apparently, then one can get $\text{rank}[(A-B)^2] = \text{rank}(A-B)$, then according to the conclusion of Lemma 2.1, $(A-B)^i$ exists.

Lemma 2.4 Let $A^2 = A, B^2 = B, and \text{rank}(A+B) \leq \text{rank}[A(I-BA) + B(I-AB)], then (A+B)^i$ exists.

Proof. One part, $\text{rank}[(A+B)^2] \geq \text{rank}[(A+B)^2(A+B-2I)] = \text{rank}[A(I-BA) + B(I-AB)] \geq \text{rank}(A+B); And another part, \text{rank}[(A+B)^2] \leq$
\( \text{rank}(A + B) \) apparently, then one can get \( \text{rank}[(A + B)^2] = \text{rank}(A + B) \), then according to the conclusion of Lemma 2.1, \( (A + B)^4 \) exists.

3. Conclusions

**Theorem.** Suppose \( M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, (A, B \in K^{n \times n}, \text{and} A^2 = A, B^2 = B) \), then (i) \( M^2 \) exists if and only if \( \text{rank}(A - B) \leq \text{rank}[A(I - BA) + B(I - AB)] \), (ii) if \( M^2 \) exists, and \( \text{rank}(A + B) \leq \text{rank}[A(I - BA) + B(I - AB)] \), then \( M^2 = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \left( \begin{pmatrix} A - B \end{pmatrix} \begin{pmatrix} X \\ (A + B)^2 \end{pmatrix} \right) \begin{pmatrix} I & 0 \\ (A + B)^2 & I \end{pmatrix}, \)

where \( X = (A - B)^2 B[I - (A + B)(A + B)^2] + [I - (A - B)(A - B)^2] B(A + B)^2 - (A - B)^2 B(A + B)^2 \).

**Proof.** (i) Proof of sufficient conditions. It is easy to prove that

\[
\text{rank}(M) = \text{rank} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \text{rank} \begin{pmatrix} A - B & 0 \\ 0 & A + B \end{pmatrix} = \text{rank}(A + B) + \text{rank}(A - B);
\]

\[
\text{rank}(M^2) = \text{rank} \begin{pmatrix} A^2 + B^2 & AB + BA \\ AB + BA & A^2 + B^2 \end{pmatrix} = \text{rank} \begin{pmatrix} A + B & AB + BA \\ AB + BA & A + B \end{pmatrix} = \text{rank} \begin{pmatrix} A + B & 0 \\ 0 & A + B - BAB - ABA \end{pmatrix} = \text{rank}(A + B) + \text{rank}[A(I - BA) + B(I - AB)].
\]

For the given condition \( \text{rank}(A - B) \leq \text{rank}[A(I - BA) + B(I - AB)] \), we can get

\[
\text{rank}(M) = \text{rank}(A + B) + \text{rank}(A - B) \leq \text{rank}(A + B) + \text{rank}[A(I - BA) + B(I - AB)] = \text{rank}(M^2) ;
\]

And with \( \text{rank}(M) \geq \text{rank}(M^2) \), we can easily obtain \( \text{rank}(M) = \text{rank}(M^2) \). Then according to the Lemma 2.1, \( M^2 \) exists.

Proof of the necessary conditions. \( M^2 \) exists, then \( \text{rank}(M) = \text{rank}(M^2) \), and then

\[
\text{rank}(A + B) + \text{rank}(A - B) = \text{rank}(A + B) + \text{rank}[A(I - BA) + B(I - AB)] \leq \text{rank}(A - B) \]

Can be proved.

(ii) with the condition \( \text{rank}(A - B) \leq \text{rank}[A(I - BA) + B(I - AB)] \), \( A + B \) exists. Similarly, with the condition \( \text{rank}(A + B) \leq \text{rank}[A(I - BA) + B(I - AB)] \) and the Lemma 2.4, \( (A + B)^4 \) exists.

For \( M = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} A - B & B \\ 0 & A + B \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \)

and \( \begin{pmatrix} A - B & B \\ 0 & A + B \end{pmatrix} \rightarrow \begin{pmatrix} A - B & 0 \\ 0 & A + B \end{pmatrix} \),

we know \( \text{rank}(M) = \text{rank}(A + B) + \text{rank}(A - B) \). And with the existence of \( (A + B)^2 \) and \( (A - B)^2 \), according to the Lemma 2.2, \( M^2 \) has the form of (1).
Corollary 3.1  suppose $M = \begin{pmatrix} A & A \\ A & A \end{pmatrix}, (A \in K^{n \times n}, and A^2 = A)$, then

(i) $M^2$ exists;

(ii) if $M^2$ exists, then $M^2 = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} (A-I)^2 & X \\ 0 & (A+I)^2 \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$.

Proof. We only need to replace $B$ with $A$ in Theorem, and easily prove that $(2A)^2$ exists, then the conclusion comes true.

Corollary 3.2  suppose $M = \begin{pmatrix} A & I \\ I & A \end{pmatrix}, (A \in K^{n \times n}, and A^2 = A)$, then (i)

$M^2$ exists; (ii) if $\text{rank}(A+I) \leq \text{rank}(A-I)$, then

$M^2 = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} A-I^2 \\ 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$,

where $X = (A-I)^2[I-(A+I)(A-I)^2] + [I-(A-I)(A-I)^2](A+I)^2 - (A-I)^2(A+I)^2$.

Proof. In Theorem we replace $B$ with $I$. For $\text{rank}(A-I) \leq \text{rank}[A(I-IA) + I(A-IA)]$ and $\text{rank}(A+I) \leq \text{rank}(A-I)$, then $(A-I)^2$ and $(A+I)^2$ exist, so the Corollary 3.2 can be proved easily.

Corollary 3.3  suppose $M = \begin{pmatrix} A & A^* \\ A^* & A \end{pmatrix}, (A \in K^{n \times n}, and A^2 = A)$, then (i)

$M^2$ exists; (ii) if $M^2$ exists, then

$M^2 = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} A-A^* \\ 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$,


Proof. In Theorem we replace $B$ with $A^*$. For $(A-A^*)$ and $(A+A^*)$ are respectively anti-Hermitian matrix and the Hermite matrix, they are unitarily similar to diagonal matrices, so $(A-A^*)^2$ and $(A+A^*)^2$ exist, then the Corollary 3.3 can be proved easily.

Corollary 3.4  suppose $M = \begin{pmatrix} A & AA^2 \\ AA^2 & A \end{pmatrix}, (A \in K^{n \times n}, and A^2 = A)$, then (i)

$M^2$ exists; (ii) if $\text{rank}(A + AA^2) \leq \text{rank}(A - AA^2)$, then

$M^2 = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} A-AA^2 \\ 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$,


Proof. In Theorem we replace $B$ with $AA^2$. For $\text{rank}(A - AA^2) = \text{rank}[(A(I- AA^2A) + AA^2(I - AA^2A)]$ and $\text{rank}(A + AA^2) \leq \text{rank}(A - AA^2)$, so $(A - AA^2)^2$ and $(A + AA^2)^2$ exist. Therefore it, then the Corollary 3.4 can be proved easily.
References

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