A Non-uniform Bound on Poisson Approximation of Empty Urn-Model via Stein-Chen Method

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Abstract

In this paper, we give a non-uniform bound for approximation of the number of empty urns with equal probability of urn, after throwing balls independently by Poisson distribution via Stein-Chen coupling method.

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1 Introduction

For the useful and classical theory of the urn models can be contributed to several fields of studies ([8], [9] and [10]), since many problems in the area of probability theory, combinatorial analysis, physical sciences, biological sciences, social sciences, computer science, and the others can be described in terms of distributing ball (objects) into specified urns (locations). In computer science, urn models are used for database performance evaluations and for modeling and analyzing algorithms. In medical science, urn model is applied to study cone ratios in human and macaque retinas. In economics, urn models are used to capture the mechanism of reinforcement learning. In communication theory, some transmission channels can be described in terms of contagion urn models. Among the most commonly encountered urn models in physics are the so called Maxwell-Boltzman, Bose-Eistein and Fermi-Dirac model.
In the usual formulation of empty urn problem, we assume that \( m \) indistinguishable balls are thrown independently into \( n \) urns that have equal probability and compute the probability of the urns that are empty after the balls have been thrown.

Let \( W \) be the number of empty urns. We will investigate the bound of the probability approximation of \( W \) by Poisson distribution.

For each \( i \in \{1, 2, 3, \ldots, n\} \), we define the indicator random variable \( X_i \), as follows:

\[
X_i = \begin{cases} 
1 & \text{if the } i\text{-th urn is empty}, \\
0 & \text{otherwise}.
\end{cases}
\]

Therefore,

\[
\mathbb{E}(X_i) = \left(1 - \frac{1}{n}\right)^m = \left(\frac{n-1}{n}\right)^m
\]

And set

\[
W = \sum_{i=1}^{n} X_i.
\]

Then \( W \) is the number of empty urns and we have \( \lambda = \mathbb{E}(W) = n \left(\frac{n-1}{n}\right)^m \).

In 1992, Barbour and Chen ([1], p.73-75) showed that the distribution of \( W \) can be approximated by Poisson distribution with parameter \( \lambda \). Here is their result.

**Theorem 1.1.** Let \( W \) be the number of empty urns. Then

\[
\sup_{A \subseteq \mathbb{N}} |P(W \in A) - \text{Poi}_\lambda(A)| \leq (1 - e^{-\lambda}) \left(\lambda - (n-1) \left(\frac{n-2}{n-1}\right)^m\right).
\]

where \( \text{Poi}_\lambda \) is Poisson distribution with parameter \( \lambda \).

In this paper, we give a non-uniform bound on Poisson approximation of the number of empty urns by using Stein-Chen coupling method which is introduced in section 2. The following are our main results.

**Theorem 1.2.** Let \( W \) be the number of empty urns. Then

\[
|P(W \in A) - \text{Poi}_\lambda(A)| \leq C_{\lambda,m,\Delta} \left(\frac{1}{n-1}\right)
\]

where \( C_{\lambda,m,\Delta} = \max\{\binom{m-1}{n-2}, \binom{m}{n-2}\} \min\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\} \),

\[
\Delta(\lambda) = \begin{cases} 
e^\lambda + \lambda - 1 & \text{if } \lambda^{-1}(e^\lambda - 1) \leq M_A, \\
2(e^\lambda - 1) & \text{if } \lambda^{-1}(e^\lambda - 1) > M_A
\end{cases}
\]
and

\[ M_A = \begin{cases} 
\max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\
\min\{w \mid w \in A\} & \text{if } 0 \not\in A 
\end{cases} \]

when \( C_w = \{0, 1, 2, \ldots, w\} \).

**Corollary 1.3.** Let \( W \) be the number of empty urns. Then

\[ \sup_{A \subseteq \mathbb{N}} |P(W \in A) - \text{Poi}_\lambda(A)| \leq \alpha_m(1 - e^{-\lambda}) \left( \frac{1}{n - 1} \right) \]

where \( \alpha_m = \max\{\binom{m}{\frac{m}{2}}, \binom{m}{\frac{m}{2}}\} \).

## 2 Poisson approximation via Stein-Chen method

The Stein’s method is an interesting method in probability theory to obtain bounds on the distance between two probability distributions. The method was originally formulated for standard normal distribution of sums of dependent random variables by Charles Stein in 1972 [4]. Further more, his basic idea was applied for other studies. In 1975, Louis Chen Hsiao Yun [7] modified Stein’s method so as to obtain approximation results for the Poisson distribution, therefore the method is often referred to as Stein-Chen method. In 1992, Barbour, Holst and Janson [2] were developed the Stein-Chen coupling method and give the fundamental result, as follow.

**Theorem 2.1.** Let \( W = \sum_{i=1}^{n} X_i, \ p_i = E(X_i) = P(X_i = 1), \ \lambda = E(W) \) and for each \( i, \ W_i \) be the random variable on the same probability space as \( W \). If the distribution of \( W \) equals to the conditional distribution of \( W - X_i \mid X_i = 1 \). Then

\[ |P(W \in A) - \text{Poi}_\lambda(A)| \leq \| g_{\lambda,A} \| \sum_{i=1}^{n} p_i E(|W - W_i|) \] (2.1)

where \( \| g_{\lambda,A} \| := \sup_{w \in A}[g_{\lambda,A}(w + 1) - g_{\lambda,A}(w)] \).

The conception in the Stein-Chen method for the Poisson distribution with parameter \( \lambda \), is given by, for any \( \lambda > 0 \) and \( A \subseteq \mathbb{N} \) there exists a function \( g_{\lambda,A} \) on \( \mathbb{N} \cup \{0\} \) satisfying Stein’s equation

\[ I_A(j) - \text{Poi}_\lambda(A) = \lambda g_{\lambda,A}(j + 1) - j g_{\lambda,A}(j), \quad j \geq 0 \] (2.2)
where \( I_A : \mathbb{N} \cup \{0\} \to \mathbb{R} \) is an indicator function.

By substituting \( j \) and \( \lambda \) in (2.2) by \( W = \sum_{i=1}^{n} X_i , \ \lambda = \text{E}(W) \) and take expectation, we have

\[
P(W \in A) - \text{Poi}_\lambda(A) = \text{E}(\lambda g_{\lambda,A}(W + 1)) - \text{E}(W g_{\lambda,A}(W)). \tag{2.3}
\]

The well known solution \( g_{\lambda,A} \) of (2.2) is of the form

\[
g_{\lambda,A}(w) = \begin{cases} (w - 1)!\lambda^{-w}e^{\lambda}\left[\mathcal{P}_\lambda(I_{A \cap C_{w-1}}) - \mathcal{P}_\lambda(I_A)\mathcal{P}_\lambda(I_{C_{w-1}})\right] & ; w \geq 1, \\
0 & ; w = 0
\end{cases}
\]

where \( \mathcal{P}_\lambda(I_A) = e^{-\lambda} \sum_{i=0}^{\infty} I_A(i) \frac{\lambda^i}{i!} \) and \( C_{w-1} = \{0, 1, 2, \ldots, w - 1\} \).

Several authors determined a bound of \( \| g_{\lambda,A} \| \). For \( A \subseteq \mathbb{N} \cup \{0\} \), in 1975, Chen [7] showed that \( \| g_{\lambda,A} \| \leq \min\{1, \lambda^{-1}\} \) and, in 1994, Janson [11] showed that

\[
\| g_{\lambda,A} \| \leq \lambda^{-1}(1 - e^{-\lambda}) \leq \min\{1, \lambda^{-1}\}. \tag{2.4}
\]

In case of non-uniform bound, in 2003, Neammanee [5] showed that \( \| g_{\lambda,A} \| \leq \min\left\{ \frac{1}{w_0}, \lambda^{-1}\right\} \) and, in 2005, Teerapabolarn and Neammanee [6] gave bound of \( \| g_{\lambda,A} \| \) where \( A = \{0, 1, \ldots, w_0\} \) in the terms of

\[
\| g_{\lambda,A} \| \leq \lambda^{-1}(1 - e^{-\lambda}) \min\left\{1, \frac{e^{\lambda}}{w_0 + 1}\right\}.
\]

In general case for any subset \( A \) of \( \{0, 1, \ldots, n\} \), in 2006, Santiwipanont and Teerapabolarn [12] gave a bound in the form of

\[
\| g_{\lambda,A} \| \leq \lambda^{-1} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A + 1}\right\} \tag{2.5}
\]

where

\[
\Delta(\lambda) = \begin{cases} e^\lambda + \lambda - 1 & \text{if } \lambda^{-1}(e^\lambda - 1) \leq M_A, \\
2(e^\lambda - 1) & \text{if } \lambda^{-1}(e^\lambda - 1) > M_A,
\end{cases}
\]

and

\[
M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\
\min\{w \mid w \in A\} & \text{if } 0 \not\in A.
\end{cases}
\]
The another part in apply Theorem 2.1 is to construct distribution of $W_i$ equals to the conditional distribution of $W - X_i | X_i = 1$ and make $E|W - W_i|$ small. For the case of $X_1, \ldots, X_n$ are independent, we let $W_i = W - X_i$. Then $E|W - W_i| = p_i$ and from (2.1), we have $|P(W \in A) - Pois(\lambda)| \leq \|g_{\lambda,A} \| \sum_{i=1}^{n} p_i^2$.

In next section, we will use Theorem 2.1 to prove our main results by constructing the random variable $W_i$.

3 Proof of Main Results

Proof of Theorem 1.2. For any $i \in \{1, 2, 3, \ldots, n\}$, we construct $W_i$ in the following way. Take those balls which have landed in the $i$-th urn, remove the $i$-th urn and throw them independently into other urns, the random variable $W_i$ is the number of empty urns after we removing the $i$-th urn which is empty. Then the distribution of $W_i$ equals to the conditional distribution $(W - X_i | X_i = 1)$, that is for $k \in \{0, 1, 2, \ldots, n-1\}$,

$$P(W_i = k) = \left(1 - \frac{k}{n-1}\right)^m = \left(\frac{n-k-1}{n-1}\right)^m$$

and

$$P(W - X_i = k | X_i = 1) = \frac{P(W = k + 1, X_i = 1)}{P(X_i = 1)} = \frac{(1 - \frac{k+1}{n})^m}{(1 - \frac{1}{n})^m} = \left(\frac{n-k-1}{n-1}\right)^m.$$ 

Thus $P(W_i = k) = P(W - X_i = k | X_i = 1)$.

We observe that in case $X_i = 1$, so we have the $i$-th urn is empty. Thus the number of urns that are empty after removing the $i$-th urn, equals to the number of the empty urns minus 1, that is $W_i = W - 1$.

For any $j \in \{1, 2, 3, \ldots, n\}$ such that $j \neq i$, we define the indicator random variable $X_{ij}$, as follow,

$$X_{ij} = \begin{cases} 
1 & \text{if the } j\text{-th urn is not empty after we throw the balls again,} \\
& \text{in which these balls exactly used to land the } i\text{-th urn before,} \\
0 & \text{otherwise.}
\end{cases}$$

In case $X_i = 0$, the number of the urns that are empty after removing the $i$-th urn and we throw them again as defined, equals to the number of the empty urns minus the sum of number of the $j$-th urn, where $j \neq i$, is empty.
in the first-throw and they are not empty after we throw them again, that is

\[ W_i = W - \sum_{j=1, j \neq i}^{n} X_j X_{ij}. \]  \hfill (3.1)

We know that

\[ E|W - W_i| = E(W - W_i)^+ + E(W - W_i)^-. \]

where \( (W - W_i)^+ = \max\{W - W_i, 0\} \) and \( (W - W_i)^- = -\min\{W - W_i, 0\} \).

Form (3) and (3.1),

- case \( X_i = 1 \) we have \( (W - W_i)^+ = 1 \) and \( (W - W_i)^- = 0 \),

- case \( X_i = 0 \) we have \( (W - W_i)^+ = \sum_{j=1, j \neq i}^{n} X_j X_{ij} \) and \( (W - W_i)^- = 0 \).

Therefore,

\[ (W - W_i)^+ \leq \sum_{j=1, j \neq i}^{n} X_j X_{ij} \quad \text{and} \quad (W - W_i)^- = 0. \]

From this fact we have

\[
E(W - W_i)^+ \leq E \left( \sum_{j=1, j \neq i}^{n} X_j X_{ij} \right) \\
= \sum_{j=1, j \neq i}^{n} E(X_j X_{ij}) \\
= \sum_{j=1, j \neq i}^{n} P(X_j = 1, X_{ij} = 1) \\
= (n - 1) \left( 1 - \frac{1}{n} \right)^m \left( 1 - \left( 1 - \frac{1}{n(n-1)} \right)^{b_i} \right) \\
\leq (n - 1) \left( 1 - \left( 1 - \frac{1}{n(n-1)} \right)^m \right)
\]

where \( b_i \) is the number of balls contained in the \( i \)-th urn.
Thus
\[
\mathbb{E}(|W - W_i|) \leq (n - 1) \left( \frac{(n(n-1))^m - (n(n-1) - 1)^m}{(n(n-1))^m} \right)
\]
\[
= (n - 1) \left( \frac{\sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} (n(n-1))^{m-k}}{(n(n-1))^m} \right)
\]
\[
\leq (n - 1) \left( \sum_{k=1}^{m} \binom{m}{k} (n(n-1))^{-k} \right)
\]
\[
\leq \alpha_m (n - 1) \left( \sum_{k=1}^{m} (n(n-1))^{-k} \right)
\]
\[
= \alpha_m (n - 1) \left( \frac{(n(n-1))^{-1} \left(1 - (n(n-1))^{-m}\right)}{1 - (n(n-1))^{-1}} \right)
\]
\[
\leq \alpha_m (n - 1) \left( \frac{1}{1 - (n(n-1))^{-1}} \right)
\]
\[
= \alpha_m \frac{1}{n - (n-1)^{-1}}
\]
\[
\leq \frac{\alpha_m}{n - 1}
\]
(3.2)

where \(\alpha_m = \max\{\binom{m}{\frac{m}{2}}, \binom{m}{\frac{m}{2}}\}\).

Hence, by (2.1), (2.5) and (3.2), we have
\[
|P(W \in A) - \text{Poi}_\lambda(A)| \leq C_{\lambda,m,A} \left( \frac{1}{n - 1} \right)
\]
where \(\lambda = n \left( \frac{n-1}{n} \right)^m\) and \(C_{\lambda,m,A} = \max\{\binom{m}{\frac{m}{2}}, \binom{m}{\frac{m}{2}}\} \min\{1, \lambda, \frac{\triangle(\lambda)}{M_{A+1}}\}\).

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**References**


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