Reformulation of Shapiro’s inequality

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Abstract

We reformulate Shapiro’s inequality with elementary mathematics and present some new Shapiro type inequalities by giving examples with an analytic proof.

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1 Introduction

In 1954 H. S. Shapiro [1] conjectured that

\[ E(x) = \sum_{k=1}^{n} \frac{x_k}{x_{k+1} + x_{k+2}} \geq \frac{n}{2} \quad (P(n)) \]  

(1)

where \( x_k \geq 0, \ x_{k+1} + x_{k+2} > 0 \) and \( x_{n+k} = x_k \) for \( k \in \mathbb{N} \). Equality occurring only if all denominators are equal.

Studies on (1) have been based on counterexamples and analytic proofs have given for small \( n \) so far. It is conjectured that (1) is true for even \( n \leq 12 \) and false for even \( n \geq 14 \) and that it is true for odd \( n \leq 23 \) and false for odd \( n \geq 25 \).

We now give a brief history of attempts on conjecture. Let

\[ \lambda(n) = \frac{1}{n} \inf_{x_1, x_2, \ldots, x_n} E(x). \quad \text{Then} \quad \lambda(n) \leq \frac{1}{2} \]  

(2)

clearly. The case for \( n = 1, 2 \) is trivial.

Several authors [4] proved that (2) is true for \( n = 3, 4, 5, 6 \). Diananda [3] proved that (2) is true for \( n \leq 6 \) different from the previous ones. Mordell [4] conjectured that (1) is false for all \( n \geq 7 \), but later [5] proved that (1) is true for \( n = 7 \). Nortover [2], acknowledged assistance from M. J. Lighthill, gave a counterexample for \( n = 20 \).
In [2, 4, 5, 6, 9, 27], it was proved that (1) is false for all even \( n \geq 14 \) and this result was also credited to Herschorn and Peck [25]. Zulauf [7, 8] proved that (1) is false for even \( n \geq 14 \) and is false for odd \( n \geq 53 \). Dojokovic [9] proved that \( P(8) \) is true. Rankin [6] proved that the inequality (1) is false for large enough \( n \).

Diananda [10] proved that (i) if \( P(m) \) is true, where \( m \) is even, then \( p(n) \) is true for all \( n \leq m \), and (ii) if \( P(m) \) is false, where \( m \) is odd, then \( p(n) \) is false for all \( n \geq m \). In the same paper a counterexample for \( P(27) \) was given and thus (2) is false for all odd \( n \geq 27 \).

Nowosad [11] analytically proved that \( P(10) \) is true. Bushell and Craven [12] also analytically proved that \( P(10) \) is true and thus is true for all \( n \leq 10 \) and gave a counterexample for \( n = 25 \). Godunova and Levin [13] verified \( P(12) \) partly analytically and partly numerically.

Recently, Bushell and Mcleod [14] proved analytically that \( P(12) \) is true.

Rankin [6, 15] gave a lower bound for \( \lambda = \lambda(n)_{n \to \infty} \geq 0.3047 \) and \( \lambda = \lambda(n)_{n \to \infty} \geq 0.330232 \) respectively. Prior to Rankin’s result, the only lower bound was known [8] for \( \lambda(24) = 0.49950317 \). Diananda [16, 17] improved lower bounds, found by Rankin, to \( \lambda = \lambda(n)_{n \to \infty} \geq 0.457107 \) and \( \lambda = \lambda(n)_{n \to \infty} \geq 0.461238 \) respectively. Zulauf [8, 18] showed that \( \lambda \leq \lambda(24) < 0.49950317 \) later [17] improved to \( \lambda \leq \lambda(24) < 0.499197 \) and also gave a counterexample for \( n = 24 \). Baston [24] obtained a lower bound which is an improvement on Rankin’s original result [15].

Drinfeld [19] prove that \( \lambda = \lambda(n)_{n \to \infty} = 0.4945668 \). An analytic result for the same bound, with some difficulties mentioned, also occurred [20].

Malcolm [23] numerically gave a counterexample for \( n = 25 \), Daykin [26] numerically showed that (1) is false for \( n = 14, 16, 25, 27, 40, 41, 50, 51, 60, 61, 110, 111 \) and gave counterexamples for \( n = 25, 111 \) and also found that \( \lambda \leq \lambda(111) < 0.49656 \). Troesch [21, 22] numerically proved that \( P(13) \) and \( P(23) \) are true.

For more sophisticated analysis and a brief history on conjecture see [11, 14, 22, 27, 29].

## 2 Main Results

Set

\[
F(k) = \frac{x_k}{x_{k+1} + x_{k+2}} > 0, \quad (3)
\]

where \( k = 1, 2, \ldots, n \). Then one writes

\[
\prod_{k=1}^{n} F(k) = \frac{x_1 x_2 x_3 \ldots x_n}{(x_2 + x_3) \ldots (x_n + x_{n+1}) (x_{n+1} + x_{n+2})}.
\]
Using the arithmetic mean and geometric mean inequality we reformulate Shapiro’s inequality in terms of given data as
\[
\sum_{k=1}^{n} F(k) \geq n \left\{ \frac{x_1 x_2 x_3 \ldots x_n}{(x_2 + x_3) \ldots (x_n + x_{n+1})(x_{n+1} + x_{n+2})} \right\}^{1/n}, \tag{4}
\]
where \(x_{n+1} = x_1\) and \(x_{n+2} = x_2\). Equality occurs if all \(x_k\)'s are equal. So we formally prove the following theorem.

**Theorem 2.1.** Let \(x_k > 0\) and \(x_{n+k} = x_k\) be for all \(k \in \mathbb{N}\). Then
\[
\sum_{k=1}^{n} F(k) \geq n \left\{ \frac{x_1 x_2 x_3 \ldots x_n}{(x_2 + x_3) \ldots (x_n + x_{n+1})(x_{n+1} + x_{n+2})} \right\}^{1/n}, \tag{5}
\]
equality occurs if all \(x_k\)'s are equal.

The following result is immediately follows from the above theorem.

**Corollary 2.2.** \(\prod_{k=1}^{n} F(k) \leq \frac{1}{2^n}\).

**Proof.** Applying \((x_k + x_{k+1}) \geq 2 \sqrt{x_k x_{k+1}}\) \((k = 1, 2, \ldots, n)\) to the denominator of (4), one obtains
\[
(x_2 + x_3) \ldots (x_n + x_{n+1})(x_{n+1} + x_{n+2}) \geq 2^n x_1 x_2 x_3 \ldots x_n - 1 x_{n+1} x_{n+2}.
\]
Therefore, \(\prod_{k=1}^{n} F(k) \leq \frac{1}{2^n}\). \(\square\)

We will consider \(x_k > 0\) in the following lemmas where \(k = 1, 2, \ldots, n\).

**Lemma 2.3.** \(E(x)\) is a homogeneous function of degree 0.

**Proof.** Proof is trivial. \(\square\)

**Lemma 2.4.** \(E(x)\) satisfies differential equation \(\sum_{k=1}^{n} x_k E_{x_k}(x) = 0\)

**Proof.** It is clear that \(E(x)\) possess continuous partial derivatives.
\[
\sum_{k=1}^{n} x_k E_{x_k}(x) = 0
\]
follows immediately from Lemma 2.3. \(\square\)

For these type properties of \(E(x)\) see [11, 14, 27].

**Lemma 2.5.** Let \((x_k + x_{k+1}) \leq 2 \max\{x_k, x_{k+1}\}\), \(x_{n+1} = x_1\) and \(x_{n+2} = x_2\) be where \(k = 1, 2, \ldots, n\). Then
\[
\prod_{k=1}^{n} F(k) \geq \frac{1}{2^n}.
\]
Proof. If \( \max\{x_k, x_{k+1}\} = x_k \) or \( x_{k+1} \) then
\[
\prod_{k=1}^{n}(x_k + x_{k+1}) \leq 2^n x_1 x_2 x_3 \ldots x_{n-1} x_n.
\]
Thus, \( \prod_{k=1}^{n} F(k) \geq \frac{1}{2^n} \).
\(\square\)

**Theorem 2.6.** If Lemma 2.5 holds then one obtains Shapiro’s inequality
\[
\sum_{k=1}^{n} F(k) \geq \frac{n}{2}.
\]

Proof. Using (4), proof follows immediately from Lemma 2.5. \(\square\)

3 Examples: Some new Shapiro type inequalities

We want to look at the following interesting identity [for proof, see [28, p.25]].
\[
\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}} \quad (n = 2, 3, \ldots).
\]

**Example 3.1.** Set
\[
F(k) = \sin\left(\frac{k\pi}{n}\right) \quad \text{where} \quad k = 1, 2, \ldots, n - 1.
\]

Then using the above identity together with (4) one gets
\[
\sum_{k=1}^{n-1} F(k) \geq (n - 1)\left(\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right)\right)^{\frac{1}{n-1}} = (n - 1)\left(\frac{n}{2^{n-1}}\right)^{\frac{1}{n-1}} \geq \frac{n - 1}{2},
\]
since \( 1 \leq n^{1/(n-1)} \leq 2 \) as \( n \) varies from 2 to \( \infty \).

**Example 3.2.** Set
\[
F(k) = \sin\left(\frac{k\pi}{n}\right) \quad \text{where} \quad k = 1, 2, \ldots, n - 1 \quad \text{and} \quad x_n = 1.
\]

Then using the above identity together with (4) one gets
\[
\sum_{k=1}^{n} F(k) \geq n\left(\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right)x_n\right)^{\frac{1}{n}} = n\left(\frac{n}{2^{n-1}}\right)^{\frac{1}{n}} = n\left(\frac{2n}{2^n}\right)^{\frac{1}{n}} \geq \frac{n}{2},
\]
since \( 1 \leq (2n)^{1/n} \leq 2 \) as \( n \) varies from 2 to \( \infty \).
References


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