A (Very) Simple Proof that $H^1(G, V) = (0)$

for a Compact, or Connected Semi-Simple Group

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Abstract

In this note we give a (very) simple proof of the known fact that the first cohomology group with coefficients in a finite dimensional real vector space $V$ of a compact, or of a connected semi-simple group $G$ must vanish.

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1 Introduction.

Here we prove that $H^1(G, V) = (0)$ for any compact, or connected semi-simple group $G$ with coefficients in a finite dimensional vector space $V$. The compact case is very well known, see for example Moskowitz, [7] pg. 334, and actually it is proved that for a compact group all the higher order cohomology groups vanish, even when $V$ is a Banach space (see [4], Theorem 6.0.3). Things are quite different when $G$ is a non compact connected semi-simple Lie group.
Here there only seem to be proofs of the following two results:

Let $G$ be a real, connected, semi-simple Lie group acting continuously on a Banach space $V$.

1. If none of the simple components is locally isomorphic to $SO_o(n, 1)$ or $SU(n, 1)$, then $H^1(G, V) = (0)$. (Erven-Kazdan [3] Chapter V).

2. If $G$ is simply connected, then $H^1(G, V) = (0)$ (S. Komy [5]).

For a counter example in the case of $SO_o(n, 1)$ (which works equally well for $SU(n, 1)$) see [4] pg. 118, or the original proof in [2].

For the reader convenience we recall the definition of the first cohomology group $H^1(G, V)$.

Let $G$ be a locally compact, second countable group and $\rho$ be a continuous representation of $G$ on a real finite dimensional vector space $V$. We will use without distinction the notations $\rho(g)(v)$, or $g.v$ ($g \in G$ and $v \in V$). Then, the first cohomology group $H^1(G, V)$ is defined as follows:

**Definition 1.** $H^1(G, V)$ is defined to be the quotient group $Z^1/B^1$, where $Z^1$ is the space of the crossed homomorphisms (or 1-cocycles)

$$\varphi : G \rightarrow V : \varphi(gh) = \varphi(g) + g\varphi(h),$$

and $B^1$ consists of those $\varphi$ (or 1-coboundaries) of the form $\varphi(g) = g.v_0 - v_0$, for some $v_0$ in $V$ and all $g$ in $G$.

Based in a geometric observation of Milnor ([6]), we shall give a very simple proof that $H^1(G, V) = (0)$ ($V = \mathbb{R}^n$) dealing with the compact and semi-simple cases simultaneously.

## 2 Main Theorem.

In fact, we have the following unifying result:

**Theorem 1.** Let $G$ be a group all of whose finite dimensional real representations are completely reducible. Then for every finite dimensional representation of $G$ on $V$, $H^1(G, V) = (0)$. 
In particular,

**Corollary 1.** If $G$ contains a connected semi-simple subgroup $H$ with $G/H$ either compact or of finite volume, then all finite dimensional real representations $\rho$ are completely reducible. (Of course if $G$ is compact, or connected semi-simple this is so. Hence, in all these cases $H^1(G,V) = (0)$).

**Proof.** Since $H$ is connected semi-simple any continuous representation is completely reducible by H. Weyl’s theorem (see e.g. [1] p. 175). Moreover, as is proved in Moskowitz [8] (Theorem 1, or Corollary 2 respectively), since $G/H$ is either compact, or of finite volume, $\rho$ must be completely reducible. $\square$

To prove Theorem 1 we need the following:

**Definition 2.** By an invertible affine transformation of a vector space $V = \mathbb{R}^n$ we mean a map $V \rightarrow V$ given by $x \mapsto Ax + b$, where $x, b \in V$, $A \in GL(V)$.

The next lemma is a slight modification of a result of Milnor (see [6] pg. 183).

**Lemma 1.** If $\rho$ is completely reducible continuous representation of $G$ by affine transformations of $V$, then $\rho$ admits a fixed point.

**Proof.** Identify the space $V$ with the hyperplane $\mathbb{R}^n \times \{1\}$ in $\mathbb{R}^{n+1}$. Now, any representation of $G$ by affine transformations of $V \times \{1\}$ extends uniquely to a linear representation of $G$ on $\mathbb{R}^{n+1}$. Indeed the map $x \mapsto Ax + b$, $x \in V$ extends to the map

$$
\begin{pmatrix}
    x \\
    1 
\end{pmatrix} \mapsto \begin{pmatrix} A & b \\
    0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\
    1 
\end{pmatrix} = \begin{pmatrix} Ax + b \\
    1 \end{pmatrix}
$$

which is linear. Since the linear subspace $\mathbb{R}^n \times \{1\}$ is invariant, by hypothesis, there exists a complementary $G$-invariant subspace $W$. Then, the intersection

$$W \cap (\mathbb{R}^n \times \{1\})$$

is a fixed point which is not the point $(0)$ since $(0)$ is not in this hyperplane. $\square$

Turning to the proof of our theorem,
Proof. Let \( \rho : G \longrightarrow GL(V) \) be a continuous linear representation of \( G \) and \( \varphi \) be a 1-cocycle. Define the affine map,

\[
\rho \varphi : G \longrightarrow \text{Aff}(V) := G \ltimes GL(V),
\]

given by

\[
\rho \varphi (g) : V \longrightarrow V \quad \text{such that} \quad \rho \varphi (g)(v) := \rho (g)(v) + \varphi (g).
\]

From the cocycle identity this map is a homomorphism. But by the Lemma 1, the affine map \( \rho \varphi \) has a fixed point. That is, there is a \( v_0 \) in \( V \) with \( \rho \varphi (g)(v_0) = v_0 \), for each \( g \in G \). Then \( \rho (g)(v_0) + \varphi (g) = v_0 \) so that \( \varphi \) is a 1-coboundary and \( H^1(G, V) = (0) \).  

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References


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