Some Properties of Operator Classes \((M, k)^*, \ A^*[k]\)

and \(k-*\) Paranormal Operator

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Abstract
In this article we have proved that every operator in \((M, k)^*\) class for \(k \geq 2\) is a \(k-*\) paranormal operator, also we give some properties about these classes. We showed that for every non-zero operators \(T_1 \in B(H_1)\) and \(T_2 \in B(H_2)\) their tensor product \(T_1 \otimes T_2\) belongs to the \(A^*[k]\) class, if and only if, \(T_1\) and \(T_2\) belong to the \(A^*[k]\) class, for \(k \geq 1\).

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1. Introduction

Let us denote by \(H\) the complex Hilbert space and \(B(H)\) the space of all bounded linear operators defined in Hilbert space \(H\). In the following we will mention some known classes of operators defined in Hilbert space \(H\). Let \(T\) be an element in the algebra of bounded operators \(B(H)\). The operator \(T\) is called quasi-normal if \(T(T^*T) = (T^*T)T\), it is hyponormal if \(T^*T \geq TT^*\), which is equivalent to the condition \(\|Tx\| \geq \|T^*x\|\), for all \(x\) in \(H\). We say that an operator \(T\) is quasi-hyponormal if the following condition: \(T^{2}T^{2} \geq (T^{*}T)^{2}\) holds and the last one is equivalent with \(\|T^{2}x\| \geq \|T^*Tx\|\), for all \(x\) in \(H\). We say that an operator \(T\) is of \((M,k)\) class if \(T^{*k}T^{k} \geq (T^{*}T)^{k}\), for \(k \geq 2\), which is equivalent to the condition \(\|T^{k}x\| \geq \|(T^{*}T)^{\frac{k}{2}}x\|\), for all \(x\) in \(H\) and \(k \geq 2\), (see [8]). It is known that the \((M,2)\) class coincides with the class of quasi-hyponormal operators. But, the class of hyponormal operators does not coincide with \((M,k)\), for any \(k\), (see [7]). We say that an operator \(T\) is of \((M,k)\)’ class if \(T^{*k}T^{k} \geq (T^{*}T)^{k}\), for \(k \geq 1\), which is equivalent to the condition \(\|T^{k}x\| \geq \|(T^{*}T)^{\frac{k}{2}}x\|\), for all \(x\) in \(H\) and \(k \geq 1\), (see [8]). It is known that the \((M,1)\)’ class coincides with the class of hyponormal operators. The operator \(T\)
is called $k^\ast$ paranormal if it satisfies the following condition $\|T^k x\| \geq \|T^\ast x\|^k$, for all unit vectors $x$ in $H$ and $k \geq 2$. The operator $T \in B(H)$ is of $A'[k]$ class if $|T^k|^2 \geq |T^\ast|^2$, for $k \geq 1$, ($k$ - an integer). The spectrum, the point spectrum, the approximate point spectrum of an operator $T$ are denoted by $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, respectively.

**Theorem A.** (Hölder-McCarthy inequality [1]). Let $A$ be a positive operator. Then the following inequalities hold for all $x$ in $H$

1. $\langle A^\ast x, x \rangle \leq \langle Ax, x \rangle^\gamma \|x\|^{2(1-r)}$, for $0 < r \leq 1$,

2. $\langle A^\ast x, x \rangle \geq \langle Ax, x \rangle^\gamma \|x\|^{2(1-r)}$, for $r \geq 1$.

**Proposition A.** [6]. Let $T_1, T_2 \in B(H_1)$, $S_1, S_2 \in B(H_2)$ be non-negative operators. If $T_1$ and $S_1$ are non–zero operators, then the following assertions are equivalent

1. $T_1 \otimes S_1 \leq T_2 \otimes S_2$,

2. There exists $c > 0$ such that $T_1 \leq cT_2$ and $S_1 \leq c^{-1}S_2$.

**Proposition B.** [3]. For every $\alpha, \beta \in C$, $T_1, T_2 \in B(H_1)$ and $S, S_1, S_2 \in B(H_2)$,

1. $\alpha \beta(T \otimes S) = \alpha T \otimes \beta S$,

2. $(T_1 + T_2) \otimes (S_1 + S_2) = T_1 \otimes S_1 + T_2 \otimes S_1 + T_1 \otimes S_2 + T_2 \otimes S_2$,

3. $(T_1 \otimes S_1)(T_2 \otimes S_2) = T_1 T_2 \otimes S_1 S_2$,
v). \((T \otimes S)^* = T^* \otimes S^*\),

vi). \(\|T \otimes S\| = \|T\| \|S\|\).

If \(T\) and \(S\) are invertible, then so is \(T \otimes S\) and

vii). \((T \otimes S)^{-1} = T^{-1} \otimes S^{-1}\).

2. Classes of operators \((M,k)^\prime\), \(A'^*[k]\) and \(k\)-*paranormal in Hilbert space

In this section we will show some properties of \((M,k)^\prime\), \(A'^*[k]\) classes and \(k\)-*paranormal operators.

**Proposition 2.1.** For each positive integer \(k \geq 1\) an operator \(T\) belongs to class \((M,k)^\prime\) if and only if

\[ T^{*k}T^k + 2\lambda(TT^*)^k + \lambda^2 T^{*k}T^k \geq 0, \]

holds for all \(\lambda \in \mathbb{R}\).

**Proof.** Let \(\lambda \in \mathbb{R}\) and \(x \in H\) be given. Then \(T \in (M,k)^\prime\), if and only if

\[
\left\| (TT^*)^{\frac{k}{2}} x \right\| \leq \left\| T^{*k}x \right\| \Leftrightarrow 4 \left\| (TT^*)^{\frac{k}{2}} x \right\|^2 - 4 \cdot \left\| T^{*k}x \right\|^2 \cdot \left\| T^{k}x \right\|^2 \leq 0
\]

\[
\Leftrightarrow \left\| T^{k}x \right\|^2 + 2\lambda \left\| (TT^*)^{\frac{k}{2}} x \right\|^2 + \lambda^2 \left\| T^{k}x \right\|^2 \geq 0
\]

\[
\Leftrightarrow \langle (T^k x, T^k x) + 2\lambda \langle (TT^*)^{\frac{k}{2}} x, (TT^*)^{\frac{k}{2}} x \rangle + \lambda^2 \langle T^k x, T^k x \rangle \rangle \geq 0
\]

\[
\Leftrightarrow \langle (T^{*k}T^k x, T^{*k}T^k x) + 2\lambda \langle (TT^*)^{k} x, x \rangle + \lambda^2 \langle T^{*k}T^k x, x \rangle \rangle \rangle \geq 0
\]
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\[ \Leftrightarrow \langle (T^{*k}T^k + 2\lambda(TT^*)^k + \lambda^2 T^{*k}T^k)x, x \rangle \geq 0 \]

\[ \Leftrightarrow T^{*k}T^k + 2\lambda(TT^*)^k + \lambda^2 T^{*k}T^k \geq 0, \]

by which the proof is completed.■

**Corollary 2.1.** If \( k = 1 \), we get the following relation \( T^*T \geq TT^* \) if and only if \( T^*T + 2\lambda TT^* + \lambda^2 T^*T \geq 0 \), for all \( \lambda \in R \), which is the definition of the hyponormal operator.

**Lemma 2.1.** If \( T \) is a bilateral weighted shift operator, with weighted sequence \( \omega_n \), \( (Te_n = \omega_n e_{n+1}) \), then it is of \( (M,k)^* \) class if and only if

\[ |\omega_n| \cdot |\omega_{n+1}| \cdots |\omega_{n+k-1}| \geq |\omega_{n-1}|^k \], for \( n \in Z \) and \( k \geq 1 \).

**Proof.** The proof follows immediately from the definition of \( (M,k)^* \) class.■

**Example 2.1.** Let \( T \in B(H) \) be a bilateral weighted shift with weighted sequence \( (\omega_n) \) given as follows

\[ \omega_n = \begin{cases} 
\frac{1}{2}, & \text{for } n \leq -1 \\
2, & \text{for } n = 0 \\
\frac{1}{2}, & \text{for } n = 1 \\
4, & \text{for } n = 2 \\
16, & \text{for } n \geq 3.
\end{cases} \]

After some calculations, we have that \( T \in (M,3)^* \), but \( T \notin (M,k)^* \), for \( k = 1,2 \) (see lemma 2.1.).

**Example 2.2.** Let \( T \in B(H) \) be a bilateral weighted shift with weighted sequence \( (\omega_n) \) given by the formula
After some calculations, it follows that 
\[ T \in (M,2)^*, \text{ but } T \notin (M,k)^*, \] for 
\[ k = 1,3 \] (see lemma 2.1.).

**Example 2.3. (See Theorem 2.3. in [10]).** Let \( T \in B(H) \) be a bilateral weighted shift with weighted sequence \( (\omega_n) \) given as follows

\[
\omega_n = \begin{cases} 
\frac{1}{3}, & \text{for } n \leq -1 \\
1, & \text{for } n = 0 \\
\frac{1}{3}, & \text{for } n = 1 \\
3, & \text{for } n = 2 \\
\frac{1}{9}, & \text{for } n = 3 \\
729, & \text{for } n \geq 4.
\end{cases}
\]

For \( n = 1 \), by Lemma 2.1, we have \( \omega_1 \cdot \omega_2 \cdot \ldots \cdot \omega_k \geq \omega_0^k \), for \( k \geq 1 \). Therefore \( T \) is in \( (M,k)^* \) class, for \( k \geq 2 \) but it is not hyponormal, because \( \omega_0 > \omega_1 \).

**Proposition 2.2.** For each positive \( k \geq 2 \), \( T \) is a \( k^* \)-paranormal if and only if

\[
T^{*k}T^k - k\lambda^{k-1}TT^* + (k-1)\lambda I \geq 0,
\]

holds for all \( \lambda > 0 \).

**Proof.** Suppose \( T \) is a \( k^* \)-paranormal and \( x \) is a unit vector in \( H \). By generalized arithmetic-geometric mean inequality, we have
\[
\frac{1}{k} \langle \lambda^{1-k} T^k x, T^k x \rangle + \frac{k-1}{k} \langle \lambda x, x \rangle \geq \langle \lambda^{1-k} T^k x, T^k x \rangle \frac{1}{k} \langle \lambda x, x \rangle \frac{k-1}{k}
\]
\[
= \lambda^{\frac{1-k}{k}} \langle T^k x, T^k x \rangle \frac{1}{k} \lambda^{\frac{k-1}{k}} \langle x, x \rangle \frac{k-1}{k}
\]
\[
= \|T^k x\|^2 \geq \|T^* x\|^2 = \langle TT^* x, x \rangle.
\]
Hence
\[
\lambda^{1-k} \langle T^{*k} T^k x, x \rangle + \lambda (k-1) \langle x, x \rangle - k \langle TT^* x, x \rangle \geq 0.
\]
\[
\langle (T^{*k} T^k - k \lambda^{1-k} TT^* + (k-1) \lambda^k I) x, x \rangle \geq 0.
\]
Thus \( T^{*k} T^k - k \lambda^{1-k} TT^* + (k-1) \lambda^k I \geq 0 \), for all \( \lambda > 0 \).

Conversely. Let \( x \in H \), \( \|x\| = 1 \) and \( \lambda = \langle T^* x, T^* x \rangle > 0 \). Then if we put \( \lambda = \langle T^* x, T^* x \rangle > 0 \) in (4) we have
\[
\langle T^k x, T^k x \rangle - k \langle T^* x, T^* x \rangle^{k-1} \langle T^* x, T^* x \rangle + (k-1) \langle T^* x, T^* x \rangle^{k} \geq 0,
\]
\[
\|T^k x\|^2 - k \|T^* x\|^2 + (k-1) \|T^* x\|^2 \geq 0
\]
\[
\|T^k x\|^2 - \|T^* x\|^2 \geq 0.
\]
Therefore \( \|T^k x\| \geq \|T^* x\| \), respectively \( T \) is a \( k - * \)-paranormal. ■

**Corollary 2.2.** If \( k = 2 \), we get the following relation \( \|T^* x\|^2 \leq \|T^2 x\| \) if and only if \( T^{*2} T^2 - 2 \lambda TT^* + \lambda^2 I \geq 0 \), for all \( \lambda > 0 \) and \( x \in H, \|x\| = 1 \), which is the definition of the \( * \)-paranormal operator.

**Proposition 2.3.** Let \( T \) be a regular \( k - * \)-paranormal operator. Then the approximate point spectrum lies in the disc.
\[ \sigma_{\text{ap}}(T) \subseteq \left\{ \lambda \in C : \frac{\|T\|}{\|T^{* - 1}\| \cdot \|T\|} \leq |\lambda| \leq \|T\| \right\}. \]

**Proof.** Suppose \( T \) is a regular \( k \)-paranormal operator, for \( k \geq 2 \). Then for every unit vector \( x \) in \( H \), we have

\[
\|x\|^k = \|T^{* - 1}T^*x\|^k \leq \|T^{* - 1}\|^k \cdot \|T^*x\|^k \leq \|T^{* - 1}\|^k \cdot \|T^*x\| \]

\[
1 \leq \|T^{* - 1}\|^k \cdot \|T^{* - 1}\| \cdot \|T^*x\| \]

\[
\|Tx\| \geq \frac{1}{\|T^{* - 1}\| \cdot \|T^{* - 1}\|} \geq \frac{1}{\|T^{* - 1}\| \cdot \|T^*\|}. \tag{5}
\]

Now, assume that \( \lambda \in \sigma_{\text{ap}} \). Then there exists a sequence \((x_n)_n\), \( \|x_n\| = 1 \), such that \( \|(T - \lambda)x_n\| \to 0 \), when \( n \to \infty \). Therefore by (5) we have

\[
\|Tx_n - \lambda x_n\| \geq \|Tx_n - |\lambda| x_n\| \geq \frac{1}{\|T^{* - 1}\| \cdot \|T^{* - 1}\|} - |\lambda|. \tag{6}
\]

Now, when \( n \to \infty \), from relation (6) we have

\[
|\lambda| \geq \frac{1}{\|T^{* - 1}\| \cdot \|T^{* - 1}\|} = \frac{\|T\|}{\|T^{* - 1}\| \cdot \|T\|},
\]

respectively

\[ \sigma_{\text{ap}}(T) \subseteq \left\{ \lambda \in C : \frac{\|T\|}{\|T^{* - 1}\| \cdot \|T\|} \leq |\lambda| \leq \|T\| \right\}. \]

Therefore the proof is completed. \( \blacksquare \)

**Corollary 2.3.** Let \( T \) be a regular \( \ast \)-paranormal operator. Then the following relation
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\[ \sigma_{\text{op}}(T) \subseteq \{ \lambda \in \mathbb{C} : \frac{\|T\|}{\|T^{-1}\|} \leq |\lambda| \leq \|T\| \} \]

holds.

**Proposition 2.4.** Let \( T \) be a regular \( k \)-paranormal operator. Then the approximate point spectrum lies in the disc

\[ \sigma_{\text{op}}(T) \subseteq \{ \lambda \in \mathbb{C} : \frac{\|T\|}{\|T^{-1}\|} \leq |\lambda| \leq \|T\| \} . \]

**Proof.** The proof is similar with the proof of the Proposition 2.3. ■

**Theorem 2.1.** If \( T \in (M,k)^* \), \( k \geq 2 \), then \( T \) is \( k \)-paranormal operator.

**Proof.** Firstly we will prove that every operator \( T \) that belongs to the \( (M,k)^* \) it belongs to \( A^*[k] \) class, also. If \( T \in (M,k)^* \) then we have

\[ |T^k|^2 = (T^{*k} T^k)^\frac{1}{k} \geq (TT^*)^{\frac{1}{k}} = TT^* = |T^*|^2 . \]

Hence \( |T^k|^2 \geq |T^*|^2 \), for \( k \geq 2 \), respectively \( T \in A^*[k] \).

In the following let \( T \in A^*[k] \), \( k \geq 2 \). Then for every unit vector \( x \in H \) we have

\[ \|T^k x\|^2 = \langle T^k x, T^k x \rangle = \langle T^{*k} T^k x, x \rangle = \langle |T^k|^2 x, x \rangle = \langle |T^k|^{2k} x, x \rangle \]

\[ \geq \langle |T^*|^2 x, x \rangle^k \quad (\text{Hölder-McCarthy inequality}) \]

\[ \geq \langle |T^*|^2 x, x \rangle = \langle TT^* x, x \rangle^k = \langle T^* x, T^* x \rangle^k = \|T^* x\|^{2k} . \]

Finally, \( \|T^k x\| \geq \|T^* x\|^{k} \), for \( k \geq 2 \), respectively \( T \) is \( k \)-paranormal, and
Corollary 2.4. Let $T \in B(H)$ be a quasi-normal operator. If $T$ is a hyponormal operator then $T$ is a $k$-paranormal.

**Proof.** The proof of this corollary is the direct consequence of the theorem 3.11 in [7] and theorem 2.1.

Corollary 2.5. If $T \in (M,k+1)$ has dense range in $H$, then it is $k$-paranormal.

**Proof.** This proof follows from theorem 3.8 in [7] and theorem 2.1.

Remark 2.1. For $k = 2$ we obtain theorem 3.7 in [7].

Theorem 2.2. Let $T$ belongs to the $A'[k]$ class, where $k \geq 1$ and let $\alpha$ be an eigenvalue of the operator $T$. Further, let $T = \begin{bmatrix} \alpha I & 0 \\ 0 & T_1 \end{bmatrix}$ be the matrix representation in $H = \ker(T - \alpha I) \oplus \ker(T - \alpha I)^\perp$, where $T_1$ is defined in the subspace $\ker(T - \alpha I)^\perp$. Then $T_1 \in A'[k]$, for $k \geq 1$.

**Proof.** Assume that $T \in A'[k]$, $k \geq 1$, then we have

$$0 \leq |T^k|^2 - |T^*|^2 = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |T_1^k|^2 \end{bmatrix} - \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |T_1^*|^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & |T_1^k|^2 - |T_1^*|^2 \end{bmatrix}.$$  

Since $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \geq 0$ if and only if $A \geq 0$ and $C \geq 0$, it follows that

$$\begin{bmatrix} 0 & 0 \\ 0 & |T_1^k|^2 - |T_1^*|^2 \end{bmatrix} \geq 0 \iff |T_1^k|^2 - |T_1^*|^2 \geq 0,$$
Hence  \[ |T^k_i|^2 \geq |T^*_{i+k}|^2, \text{ for } k \geq 1, \text{ respectively } T_i \in A^*[k]. \]

Let \( H_1 \) and \( H_2 \) denotes the Hilbert spaces. For given non-zero operators \( T_i \in B(H_1) \) and \( T_2 \in B(H_2) \), \( T_i \otimes T_2 \) denotes the tensor on the product space \( H_1 \otimes H_2 \). Tensor product of two non-zero operators satisfy the following equalities

\[ a) (T_i \otimes T_2)^*(T_i \otimes T_2) = T_i^* T_1 \otimes T_2^* T_2 \]
\[ b) |T_i \otimes T_2|^p = |T_1|^p \otimes |T_2|^p \] for any positive real number \( p \).

**Theorem 2.3.** Let \( T_i \in B(H_1) \) and \( T_2 \in B(H_2) \) are non-zero operators. Then \( T_i \otimes T_2 \) belongs to the \( A^*[k] \) class, if and only if, \( T_i \) and \( T_2 \) belong to the \( A^*[k] \) class, for \( k \geq 1 \).

**Proof.** Let we suppose that \( T_i \) and \( T_2 \) belong to the \( A^*[k] \) class for \( k \geq 1 \). Then by Proposition B and from the properties of the tensor product of operators we have

\[ \left|(T_i \otimes T_2)^k\right|^2 = \left|T_i^k \otimes T_2^k\right|^2 = \left|T_i^k\right|^2 \otimes \left|T_2^k\right|^2 \leq \left|T_i^* \otimes T_2^*\right|^2 \]

Thus \( \left|(T_i \otimes T_2)^k\right|^2 \geq \left|(T_i \otimes T_2)^*\right|^2 \), respectively \( T_i \otimes T_2 \) belongs to the \( A^*[k] \) class.

**Conversely.** Assume that \( T_i \otimes T_2 \) belongs to the \( A^*[k] \) class, then by
Proposition B it follows that

\[ |T_1^*|^2 \otimes |T_2^*|^2 = |T_1^* \otimes T_2^*|^2 = |(T_1 \otimes T_2)^T|^2 \leq |(T_1 \otimes T_2)^T|^2 = |T_1^\kappa \otimes T_2^\kappa|^2. \]

Now by Proposition A, there exists a positive real number \( c \) such that

\[ |T_1^*|^2 \leq c|T_1^\kappa|^2 \quad \text{and} \quad |T_2^*|^2 \leq c^{-1}|T_2^\kappa|^2. \quad (7) \]

Hence, by Theorem A for every unit vector \( x \in H_1 \), we have

\[
\|T_1^*\|^2 = \sup_{H_1=1} \langle T_1^* x, T_1^* x \rangle = \sup_{|H|^1} \langle T_1^* x, T_1^* x \rangle \\
= \sup_{|H|^1} \langle T_1^* x, x \rangle \leq \sup_{|H|^1} \langle c T_1^\kappa \ | T_1^* x, x \rangle \\
= c \sup_{|H|^1} \langle T_1^\kappa | T_1^* x, x \rangle \leq c \sup_{|H|^1} \langle T_1^\kappa | x, x \rangle \quad (7) \\
= c \sup_{|H|^1} \langle T_1^\kappa | T_1^* x T_1^* x \rangle = c \|T_1^\kappa\| \leq c \|T_1\| = c \|T_1^*\|^2.
\]

Hence, \( \|T_1^*\| \leq \sqrt{c} \|T_1^*\| \) in \( H_1 \). Similarly we prove that \( \|T_2^*\| \leq \sqrt{c^{-1}} \|T_2^*\| \) in \( H_2 \). Finally, from the above two inequalities and from inequalities (7), it follows that \( c = 1 \) and therefore \( T_1, T_2 \in A^*[k] \). ■

**REFERENCES**


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