Remarks on E-Order-Preserving Transformation Semigroups

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Abstract

Let $T(X)$ be the full transformation semigroup on a set $X$. For a partially ordered set $X$, let $E$ be an arbitrary equivalence relation on $X$. We consider a subsemigroup of $T(X)$ defined by

$$T_{EO}(X) = \{ \alpha \in T(X) : \forall x, y \in X, (x, y) \in E, x \leq y \Rightarrow (x\alpha, y\alpha) \in E, x\alpha \leq y\alpha \}$$

and call it the $E$-order-preserving transformation semigroup on $X$. The purpose of this paper is to investigate relationships between $T_{EO}(X)$ and some subsemigroups of $T(X)$.

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1 Introduction

Let $T(X)$ denote the semigroup of transformations from a set $X$ into itself under composition of mappings. We call $T(X)$ the full transformation semigroup on $X$. Its subsemigroups of $T(X)$ have been widely investigated. For examples, Pei [1] has introduced a family of subsemigroups of $T(X)$ defined by

$$T_E(X) = \{ \alpha \in T(X) : \forall x, y \in X, (x, y) \in E \Rightarrow (x\alpha, y\alpha) \in E \}$$

where $E$ is an arbitrary equivalence relation on $X$. When $(X, \leq)$ is a partially ordered set, Saitô et al. [2] have considered a family of subsemigroups of $T(X)$ as follows:

$$T_\sigma(X) = \{ \alpha \in T(X) : \forall x, y \in X, x \leq y \Rightarrow x\alpha \leq y\alpha \}.$$
In this paper the set $X$ under consideration is a partially ordered set with $E$ an arbitrary equivalence relation on $X$. We define a family of subsemigroups of $T(X)$ as follows:

$$T_{EO}(X) = \{ \alpha \in T(X) : \forall x, y \in X, (x, y) \in E, x \leq y \Rightarrow (x\alpha, y\alpha) \in E, x\alpha \leq y\alpha \}. $$

Then $T_E(X) \cap T_O(X) \subseteq T_{EO}(X)$.

In this paper, we consider relationships between $T_{EO}(X)$, $T_E(X)$ and $T_O(X)$.

Following the usual terminology for a partially ordered set $X$, let us say that $a, b \in X$ are comparable if either $a \leq b$ or $b \leq a$, and incomparable if neither of these holds.

A family $\pi = \{ A_i : i \in I \}$ of nonempty subsets of $X$ is said to form a partition of $X$ if $\bigcup \pi = X$ and for all $i, j \in I$, either $A_i = A_j$ or $A_i \cap A_j = \emptyset$.

## 2 Main Results

In this section, we investigate the condition under which subsemigroups of $T(X)$ are related.

**Proposition 2.1.** Let $X$ be a partially ordered set and $E$ an arbitrary equivalence relation on $X$. Then $T_{EO}(X) = T_O(X)$ if and only if $\bigcup K \subseteq E$ where $K = \{ C \times C : C \text{ is a subchain of } X \}$.

**Proof.** Suppose that there exists $(a, b) \in \bigcup K$ such that $(a, b) \notin E$. Then $a \in A$ and $b \in B$ for some $A, B \in X/E$. Since $(a, b) \in \bigcup K$, $a, b \in C$ for some subchain $C$ of $X$. Define $\alpha \in T(X)$ by

$$ x\alpha = \begin{cases} a & \text{if } x \in B; \\ b & \text{otherwise.} \end{cases} $$

Let $x, y \in X$ be such that $x \leq y$ and $(x, y) \in E$. By definition of $\alpha$, we deduce that

$$(x\alpha, y\alpha) = \begin{cases} (a, a) \in E & \text{if } x, y \in B; \\ (b, b) \in E & \text{otherwise.} \end{cases} $$

It follows that $\alpha \in T_{EO}(X)$. Since $a$ and $b$ are comparable, we may assume that $a < b$. Then we have $a\alpha = b \not\leq a = b\alpha$. Hence $\alpha \notin T_O(X)$.

Conversely, assume that $\bigcup K \subseteq E$ where $K = \{ C \times C : C \text{ is a subchain of } X \}$. To show that $T_{EO}(X) = T_O(X)$, let $\alpha \in T_{EO}(X)$ and $a, b \in X$ with $a \leq b$.

Thus $a, b \in C$ for some subchain $C$ of $X$. By assumption, we have $(a, b) \in E$. Since $\alpha \in T_{EO}(X)$, $aa \leq ba$. Hence $\alpha \in T_O(X)$. Next, let $\alpha \in T_O(X)$ and $x, y \in X$ with $(x, y) \in E$ and $x \leq y$. Since $\alpha \in T_O(X)$, $x\alpha \leq y\alpha$ which implies that $(x\alpha, y\alpha) \in C$ for some subchain $C$ of $X$. It follows by assumption that $(x\alpha, y\alpha) \in E$. Hence $\alpha \in T_{EO}(X)$. □
Proposition 2.2. Let $X$ be a partially ordered set and $E$ an arbitrary equivalence relation on $X$. Then $T_{EO}(X) = T(X)$ if and only if for every two distinct $a, b$ in $X$, $(a, b) \in E$ implies that $a$ and $b$ are incomparable.

Proof. Suppose that there exist distinct elements $a, b$ in $X$ such that $(a, b) \in E$ and $a$ and $b$ are comparable. We may assume that $a < b$. Define $\beta \in T(X)$ by

$$x\beta = \begin{cases} a & \text{if } x = b; \\ b & \text{otherwise.} \end{cases}$$

By definition of $\beta$, we then have $a\beta = b \not\leq a = b\beta$. This means that $\beta \notin T_{EO}(X)$.

Conversely, assume that for every two distinct $a, b$ in $X$, $(a, b) \in E$ implies that $a$ and $b$ are incomparable. Let $\alpha \in T(X)$ and $x, y \in X$ with $(x, y) \in E$ and $x \leq y$. We deduce that $x = y$ which implies that $(x\alpha, y\alpha) \in E$ and $x\alpha \leq y\alpha$. Therefore $\alpha \in T_{EO}(X)$. \hfill \qed

Corollary 2.3. Let $X$ be a partially ordered set and $E$ an arbitrary equivalence relation on $X$.

(1) If $E = X \times X$, then $T_{EO}(X) = T_O(X)$ and $T_E(X) = T(X)$.

(2) If $E = I_X$, then $T_E(X) = T_{EO}(X) = T(X)$.

Theorem 2.4. Let $X$ be a partially ordered set and $E$ an arbitrary equivalence relation on $X$. If $T_{EO}(X) \subseteq T_E(X)$, then

(1) $E = X \times X$ or

(2) for each $A \in X/E$ and arbitrary partition $\{P, Q\}$ of $A$, there exist $x \in P, y \in Q$ such that $x$ and $y$ are comparable.

Proof. Suppose that $E \neq X \times X$ and (2) is not true. Then there exists $A \in X/E$ and a partition $\{P, Q\}$ of $A$ such that $a$ and $b$ are incomparable for all $a \in P, b \in Q$. Since $E \neq X \times X$, choose $B \in X/E$ such that $B \neq A$ and fix $b \in B$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b & \text{if } x \in P; \\ x & \text{otherwise.} \end{cases}$$

To show that $\alpha \in T_{EO}(X)$, let $x, y \in X$ be such that $(x, y) \in E$ and $x \leq y$. Hence $x, y \in D$ for some $D \in X/E$.

Case 1. $D \neq A$. Then $x, y \notin P$. By definition of $\alpha$, $x\alpha = x$ and $y\alpha = y$. Hence $(x\alpha, y\alpha) \in E$ and $x\alpha \leq y\alpha$.

Case 2. $D = A$. Since $x \leq y$ and $\{P, Q\}$ is a partition of $A$, either $x, y \in P$ or $x, y \in Q$. This implies that $(x\alpha, y\alpha) \in E$ and $x\alpha \leq y\alpha$.

It follows by two cases that $\alpha \in T_{EO}(X)$. Notice that for any $x \in P$ and $y \in Q$, $(x, y) \in E$ but $(x\alpha, y\alpha) = (b, y) \not\in E$. Therefore $\alpha \notin T_E(X)$. \hfill \qed
Theorem 2.5. Let \( X \) be a partially ordered set and \( E \) an arbitrary equivalence relation on \( X \). Suppose that for every \( A \in X/E \) and \( x, y \in A \), there exist subchains \( C_1, C_2, C_3, \ldots, C_n \) of \( A \) for some positive integer \( n \) such that \( x \in C_1 \), \( y \in C_n \) and \( C_i \cap C_{i+1} \neq \emptyset \) for all \( i = 1, 2, \ldots, n-1 \). Then \( T_{EO}(X) \subseteq T_E(X) \).

Proof. Suppose that for every \( A \in X/E \) and \( x, y \in A \), there exist subchains \( C_1, C_2, C_3, \ldots, C_n \) of \( A \) for some positive integer \( n \) such that \( x \in C_1 \), \( y \in C_n \) and \( C_i \cap C_{i+1} \neq \emptyset \) for all \( i = 1, 2, \ldots, n-1 \). Let \( \alpha \in T_{EO}(X) \) and \( (x, y) \in E \). Hence \( x, y \in A \) for some \( A \in X/E \). It follows by assumption that there exist subchains \( C_1, C_2, C_3, \ldots, C_n \) of \( A \) for some positive integer \( n \) such that \( x \in C_1 \), \( y \in C_n \) and \( C_i \cap C_{i+1} \neq \emptyset \) for all \( i = 1, 2, \ldots, n-1 \). Choose \( c_i \in C_i \cap C_{i+1} \) for all \( i = 1, 2, \ldots, n-1 \). Since \( x, c_1 \in C_1 \), \( x \) and \( c_1 \) are comparable. Assume that \( x \leq c_1 \). By \( \alpha \in T_{EO}(X) \), we deduce that \( (x\alpha, c_1\alpha) \in E \). For each \( i = 1, 2, \ldots, n-1 \), we have \( c_i, c_{i+1} \in C_{i+1} \). We may assume that \( c_i \leq c_{i+1} \). Since \( (c_i, c_{i+1}) \in E \) and \( \alpha \in T_{EO}(X) \), \( (c_i\alpha, c_{i+1}\alpha) \in E \). Similarly, we have that \( (c_n\alpha, y\alpha) \in E \). It follows by transitive of \( E \) that \( (x\alpha, y\alpha) \in E \). This proves that \( \alpha \in T_E(X) \). \( \square \)

Example 1. Let \( X = \{a_1, a_2, a_3, b\} \) and \( E = \{a_1, a_2\} \times \{a_1, a_2\} \cup \{a_3, b\} \times \{a_3, b\} \). We define \( \leq \ = \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_1, a_2), (a_1, a_3), (a_2, a_3), (b, b)\} \). Then \( X \) is a partially ordered set and \( E \) is an equivalence relation on \( X \). Define \( \alpha, \beta, \delta \in T(X) \) by

\[
x\alpha = \begin{cases} 
a_3 & \text{if } x = a_1; 
b & \text{if } x = a_2; 
x & \text{otherwise,}
\end{cases}
\]

\[
x\beta = \begin{cases} 
a_2 & \text{if } x = a_1; 
a_3 & \text{if } x = a_2; 
x & \text{otherwise}
\end{cases}
\]

and

\[
x\delta = \begin{cases} 
a_1 & \text{if } x = a_3; 
x & \text{otherwise.}
\end{cases}
\]

It is easy to verify that \( \alpha \in T_E(X) \setminus (T_O(X) \cup T_{EO}(X)) \), \( \beta \in T_O(X) \setminus (T_E(X) \cup T_{EO}(X)) \) and \( \delta \in T_{EO}(X) \setminus (T_O(X) \cup T_E(X)) \).

References


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