The Combinatorics of Piecewise Linear Manifolds by Colored Graphs

Alberto Cavicchioli and Fulvia Spaggiari

Dipartimento di Matematica
Università di Modena e Reggio Emilia
Via Campi 213/B, 41100 Modena, Italy
cavicchioli.alberto@unimo.it, spaggiari.fulvia@unimo.it

Abstract

Crystallization theory is a combinatorial representation of piecewise-linear (closed connected) manifolds of arbitrary dimension. This theory differs from the most important representation methods for triangulated manifolds as for example Heegaard splittings, standard spines, surgery along framed links and branched coverings, which work well in dimension less than or equal four. Crystallizations form a particular class of edge-colored multigraphs arising from combinatorial triangulations of manifolds which are minimal with respect to the number of vertices. Classical results and techniques on crystallizations are reviewed from a graph-theoretical point of view, especially to pay attention to certain new combinatorial invariants as regular genus, complexity and average order. These invariants are shown to be related with the topology of manifolds. Several open problems and conjectures concerning them complete the survey paper.

Mathematics Subject Classification: 57M15, 57Q15, 05C10

Keywords: Pseudodissection, contracted triangulation, colored graph, crystallization, orientability, connected sum, homotopy, homology, regular genus, complexity, average order
1. Pseudodissections and Contracted Triangulations

All considered (compact connected) spaces and maps belong to the piecewise linear (PL) category (see for example [47]). The prefix PL will always be omitted. For simplicity of the exposition, we shall assume that any manifold has no boundary. Two cell complexes are said to be abstractly isomorphic if there exists a bijection between them preserving the face incidence relation. By [36], p.49, an \(n\)-dimensional pseudocomplex (briefly, an \(n\)-pseudocomplex) is an \(n\)-dimensional principal cell complex in which every \(h\)-cell, considered with all its faces, is abstractly isomorphic with the complex underlying a standard \(h\)-simplex. For this reason, we shall also call \(h\)-simplex (resp. vertex) each \(h\)-cell (resp. zero-cell) of a pseudocomplex. Note that even if the space underlying a pseudocomplex is a closed manifold, the usually defined stars and links of simplexes are not necessarily cells and spheres. But there is a minimal set of severings on these stars and links, which makes them cells and spheres. The so modified stars and links are said to be disjoined, and the precise definitions are given as follows. Let \(\sigma\) be a simplex of an \(n\)-pseudocomplex \(K\). Then the disjoined star \(\text{std}(\sigma, K)\) of \(\sigma\) in \(K\) is defined as the disjoint union of the \(n\)-simplexes of \(K\) containing \(\sigma\) with re-identification of the \((n-1)\)-faces containing \(\sigma\) and of their faces. The disjoined link \(\text{lkd}(\sigma, K)\) of \(\sigma\) in \(K\) is the subcomplex of \(\text{std}(\sigma, K)\) formed by all simplexes which do not intersect \(\sigma\).

**Theorem 1.1** An \(n\)-dimensional pseudocomplex \(K\) is a closed connected combinatorial \(n\)-manifold if and only if the disjoined link of every \(h\)-simplex in \(K\) is a combinatorial \((n-h-1)\)-sphere (and hence the disjoined stars of every \(h\)-simplex is a combinatorial \(n\)-cell).

An \(n\)-pseudocomplex with exactly \(n+1\) vertices is called a contracted \(n\)-complex. A pseudodissection (resp. contracted triangulation) of an \(n\)-dimensional polyhedron \(P\) is a pair \((K, f)\), where \(K\) is an \(n\)-pseudocomplex (resp. contracted \(n\)-complex) and \(f : |K| \to P\) is a homeomorphism. As usual, \(|K|\) denotes the polyhedron underlying \(K\). The following is a famous theorem of Pezzana [45][46].

**Theorem 1.2** (Existence). Every closed connected piecewise-linear \(n\)-dimensional manifold admits a contracted triangulation.

**Proof.** Let \(M\) be a closed connected (PL) \(n\)-manifold. We can always construct \(M\) by using a single polyhedral \(n\)-cell \(B_0\) whose finitely many boundary \((n-1)\)-faces are glued together in pairs. In other words, \(M\) is the quotient
space obtained from $B_0$ by a pairwise identification of its boundary $(n - 1)$–faces (i.e., a pairing on $\partial B_0$). Let $v_0$ be an interior point of $B_0$. Starring from $v_0$ onto the simplexes of $\partial B_0$ and doing the previous pairing on $\partial B_0$, we get a new pseudodissection $K_0$ of $M$. By construction, all $n$–simplexes of $K_0$ have the same vertex $v_0$. By induction, let $K_h$, $0 \leq h < n$, be a pseudodissection of $M$ whose $n$–simplexes have the same vertices $v_0, \ldots, v_h$. Let us consider in every $n$–simplex of $K_h$ the $(n - h - 1)$–face opposite to that generated by the vertices $v_0, \ldots, v_h$. There is an ordering of the disjoined stars of such $(n - h - 1)$–faces, say, $std(\sigma_{1}^{n-h-1}, K_h), \ldots, std(\sigma_{\alpha_h}^{n-h-1}, K_h)$, such that each $std(\sigma_{i}^{n-h-1}, K_h)$ has a boundary $(n - 1)$–face which is equivalent (in the identification system defining the manifold) to a boundary $(n - 1)$–face of $std(\sigma_{j}^{n-h-1}, K_h)$ for some $j < i$. Attaching these disjoined stars along the chosen equivalent boundary $(n - 1)$–faces yields a single polyhedral $n$–cell $B_h$. Moreover, there is a pairing of the boundary $(n - 1)$–faces of $B_h$ which produces $M$ as a quotient space. Let $v_{h+1}$ be an interior point of $B_h$. Starring from $v_{h+1}$ onto the simplexes of $\partial B_h$ and doing the mentioned pairing on $\partial B_h$, we obtain a new pseudocomplex $K_{h+1}$ of $M$. By construction, all $n$–simplexes of $K_{h+1}$ have the same vertices $v_0, \ldots, v_h, v_{h+1}$. The pseudocomplex $K_n$ is a contracted triangulation of $M$. □

The theorem of Pezzana can be extended to manifolds with connected boundary and to more general spaces (some of them having also isolated singularities) [12][16].

An atlas for a contracted triangulated $n$–manifold can be immediately obtained by taking the interiors of the stars of the $n + 1$ vertices as open charts. Such an atlas satisfies the following properties:

i) Each chart is an open $n$–cell;

ii) The intersection of any number of charts is a disjoint union of open cells.

Moreover, it is minimal in the set of atlases which satisfy i) and ii). The minimality follows by using the homology exact sequence of Mayer-Vietoris, as shown in [45] (see also [46]).

2. Colored graphs and Crystallizations

Closely related to pseudocomplexes and contracted triangulations is the notion of colored graph. Let $\Delta_n$ be the set $\{0, 1, \ldots, n\}$. We shall call $\Delta_n$
the color set and its elements the colors. An \((n+1)\)-colored graph is a pair \((G, c_G)\), where \(G = (V(G), E(G))\) is a finite nonoriented multigraph, regular of degree \(n+1\), and \(c_G: E(G) \to \Delta_n\) is a proper edge-coloration, that is, \(c_G(e) \neq c_G(f)\) for any pair of adjacent edges \(e, f \in E(G)\). Every \((n+1)\)-colored graph \((G, c_G)\) represents a compact homogeneous \(n\)-dimensional pseudo-complex \(K(G)\) constructed as follows. Take an \(n\)-simplex \(\sigma^n(x)\) for each vertex \(x \in V(G)\), and label its vertices by the colors of \(\Delta_n\). If \(x, y \in V(G)\) are joined by an \(i\)-colored edge, then identify the \((n-1)\)-faces of \(\sigma^n(x)\) and \(\sigma^n(y)\) opposite to the vertices labeled by \(i\) so that equally labeled vertices are identified. Actually, the multigraph \(G\) turns out to be the 1-skeleton of the dual cellular complex of \(K(G)\). For any subset \(B \subset \Delta_n\) of cardinality \(h+1\), \(K(G)\) has as many \(h\)-simplexes with vertices colored by \(B\) as the connected components of \(G_{\hat{B}}\), where \(\hat{B} = \Delta_n \setminus B\). The theorem of Pezzana permits to represent any closed connected \(n\)-manifold by a contracted \((n+1)\)-colored graph related with a contracted triangulation of it. In fact, let \(K\) be a contracted triangulation of a closed connected \(n\)-manifold \(M\). Then we color the vertices of any \(n\)-simplex of \(K\) by the elements of \(\Delta_n\). Each \((n-1)\)-simplex of \(K\) can be labeled by the color of the vertex not belonging to it. Then the 1-skeleton of the dual cellular complex of \(K\) is a graph \(G\) (with possible multiple edges), regular of degree \(n+1\). The edges of \(G\) inherit the colors of the dual \((n-1)\)-simplexes of \(K\). So we obtain a contracted \((n+1)\)-colored graph \((G, c_G)\) such that \(K(G) = K\). Such a graph is called a crystallization of \(M\), and we say that \((G, c_G)\) represents \(M\). A lot of information on crystallizations can be found in the survey papers [3][4][7][15][23][32][41][53].

**Theorem 2.1** An \((n+1)\)-colored graph \((G, c_G)\) is a crystallization of a closed connected \(n\)-manifold if and only if every partial subgraph \(G_i\) is connected and represents the \((n-1)\)-sphere, for every \(i \in \Delta_n\).

Of course, a closed connected \(n\)-manifold can be represented by many different crystallizations. Any two colored graphs (in particular, crystallizations)
representing the same manifold are proved to be joined by a finite sequence of elementary moves [29]. To describe them we need the concept of dipole in a colored graph. Let \((G, cG)\) be an \((n + 1)\)-colored graph which admits a partial subgraph \(\Theta\), formed by two vertices \(x\) and \(y\), joined by \(h\) edges, \(1 \leq h \leq n\), labeled by the colors \(i_1, \ldots, i_h \in \Delta_n\). If \(B = \Delta_n \setminus \{i_1, \ldots, i_h\}\), let \(C_B(x)\) and \(C_B(y)\) denote the connected components of \(G_B\) containing \(x\) and \(y\), respectively. We say that \(\Theta\) is a dipole of type \(h\) if \(C_B(x)\) is different from \(C_B(y)\). Cancelling \(\Theta\) means that (1) replace in \(G\) the components \(C_B(x)\) and \(C_B(y)\) by their connected sum with respect to \(x\) and \(y\), and (2) leave unchanged the edges colored \(i_1, \ldots, i_h\) which are not incident to \(x\) and \(y\). Adding \(\Theta\) means the inverse process.

**Theorem 2.2 (Equivalence).** [29] Two colored graphs (or crystallizations) represent homeomorphic manifolds if and only if one can be transformed into the other by a finite sequence of cancelling and/or adding dipoles.

**Proof.** Let \((G, c)\) and \((G', c')\) be \((n + 1)\)-colored graphs which represent the \(n\)-manifolds \(M\) and \(M'\). Let us assume that \(G'\) is obtained from \(G\) by cancelling a dipole \(\Theta\) as above. Let \(\sigma^n(x)\) and \(\sigma^n(y)\) denote the \(n\)-simplexes in \(K = K(G)\) represented by the vertices \(x\) and \(y\) of \(\Theta\). Then \(\sigma^n(x)\) and \(\sigma^n(y)\) have exactly \(h\) common \((n - 1)\)-faces, each one of them corresponding to a color of the set \(\{i_1, \ldots, i_h\}\). The union of such common faces is a combinatorial \(n\)-cell contained in \(K\). Let \(\sigma^{n-h}\) be the \((n - h)\)-face of \(\sigma^n(x)\) and \(\sigma^n(y)\) which is the intersection of such common faces. Then \(\sigma^{n-h}\) corresponds to a connected component of the partial subgraph \(G_{\{i_1, \ldots, i_h\}}\), namely the dipole \(\Theta\). If \(\sigma^{h-1}(x)\) and \(\sigma^{h-1}(y)\) are the \((h - 1)\)-faces of \(\sigma^n(x)\) and \(\sigma^n(y)\) opposite to \(\sigma^{n-h}\), then \(\sigma^{h-1}(x)\) is different from \(\sigma^{h-1}(y)\) since \(C_B(x) \neq C_B(y)\). Cancelling \(\Theta\) from \(G\) means topologically (1) delete the interiors of \(\sigma^n(x)\) and \(\sigma^n(y)\) in \(K\), (2) pairwise identify the \((n - 1)\)-simplexes (and of all their faces) of \(\sigma^n(x)\) and \(\sigma^n(y)\) which are opposite to vertices of \(\sigma^n(x)\) and \(\sigma^n(y)\), equally colored in \(B\). This construction corresponds to the collapsing of the \(n\)-cell \(\sigma^n(x) \cup \sigma^n(y)\) to a point. So the resulting manifolds are homeomorphic, and the sufficiency is proved. Conversely, let now \((G, c)\) and \((G', c')\) be \((n + 1)\)-colored graphs which represent the same \(n\)-manifold \(M\). The pseudocomplexes \(K = K(G)\) and \(K' = K(G')\) are in stellar equivalence (see for example [47]) since one can take them to their barycentric subdivisions by elementary stellar moves. But, the first barycentric subdivision \(SdK\) of any homogeneous \(n\)-dimensional pseudocomplex \(K\) is colorable on the vertices by \(n + 1\) colors. This means that
every $n$–simplex of $SdK$ contains differently colored vertices. The coloring
is constructed as follows. A vertex of $SdK$ has color $i \in \Delta_n$ if it is the
barycenter of an $i$–simplex of $K$. So $SdK$ can be completely represented by an
$(n+1)$–colored graph by reversing the construction described at the beginning
of the section. Of course, $K$ and $K'$ are in stellar equivalence if and only
if one can be obtained from the other by adding and/or collapsing a finite
number of combinatorial $n$–cells. These operations can always be done inside
pseudocomplexes which are colorable on the vertices by $n + 1$ colors (passing
through their barycentric subdivisions if necessary). Hence they correspond
to adding and/or cancelling dipoles on the colored graphs which represent the
considered colorable pseudocomplexes.

\section*{3. Orientability and Connected Sum}

A (multi)graph $G$ is said to be bipartite if the vertex set $V(G)$ can be
partitioned into two subsets $V_1$ and $V_2$ such that every edge of $G$ joins $V_1$ with
$V_2$.

\textbf{Theorem 3.1} [17] Let $(G, c_G)$ be a crystallization of a closed connected
(PL) $n$–manifold $M$. Then $M$ is orientable if and only if $G$ is a bipartite graph.

\textbf{Proof.} Let $K = K(G)$ be the contracted triangulation of $M$, and \{\(v_i : i \in \Delta_n\}\)
the vertex-set of $K$. It is known that $K$ is orientable if and only if its $n$–
simplexes can be oriented so that any two of them induce opposite orientations
on common $(n - 1)$–faces. Since every $n$–simplex of $K$ has vertices $v_0, \ldots, v_n$,
itits orientation is given by the equivalence class of even resp. odd permutations
of vertices. Then $K$ is orientable if and only if there is an orientation of
the $n$-simplexes satisfying the following condition. If an $n$–simplex of $K$ is
oriented by $[v_0, \ldots, v_n]$, then the orientation of all adjacent $n$–simplexes must
be $-[v_0, \ldots, v_n]$. This is equivalent to the fact that all cycles of $G$ are even,
i.e., the graph is bipartite.

Let $M$ and $M'$ be two closed connected (PL) $n$–manifolds, and $(G, c)$
and $(G', c')$ crystallizations of them. A crystallization for the connected sum
$M \# M'$ can be obtained as follows. Match arbitrarily the colors of $G$ with
those of $G'$. Then take off arbitrarily a vertex from either graph, $v \in V(G)$
and $v' \in V(G')$. Finally, past together the free edges with colors corresponding
in the matching. This yields the requested crystallization, denoted by
$G \#_{v,v'} G'$. In fact, the connected sum $M \# M'$ can be performed by hollowing
out the two $n$–simplexes $\sigma \in K(G)$ and $\sigma' \in K(G')$ represented by the
deleted vertices \( v \in V(G) \) and \( v' \in V(G') \), respectively. The two permutation classes of the matching correspond to an orientation-preserving resp. orientation-reversing homeomorphism of the boundaries \( \partial \sigma \) and \( \partial \sigma' \). If both \( M \) and \( M' \) are orientable, there are two (possibly non homeomorphic) connected sums, depending on how the manifolds are oriented. To obtain the orientable connected sum we can proceed as below.

**Theorem 3.2** Let \( M \) and \( M' \) be orientable closed connected (PL) \( n \)-manifolds, and \((G, c)\) and \((G', c')\) two crystallizations of them. Consider a partition of vertices of \( G \) and \( G' \) into white and black vertices so that any edge has ends in different classes. Then the orientable connected sum of \( M \) and \( M' \) is represented by a crystallization \( G \#_{v,v'} G' \) where \( v \) is a white vertex of \( G \) and \( v' \) is a black vertex of \( G' \).

**4. Homotopy**

The encoding of a closed \( n \)-manifold \( M \) by a crystallization \((G, c)\) provides easy algorithms to deduce a finite presentation \( \langle X | R \rangle \) of the fundamental group \( \pi_1(M) \). We discuss three such algorithms.

**Algorithm 1** [33]. Choose two colors \( \alpha \) and \( \beta \) in \( \Delta_n \), and let \( X = \{x_1, \ldots, x_p\} \) be the set of all connected components, but one, of the partial \((n-1)\) subgraph obtained by deleting those colors (the missing component can be chosen arbitrarily). The connected components of the complementary \( 2 \)-subgraph are simple cycles, with edges alternatively colored by \( \alpha \) and \( \beta \) (briefly, called \( \{\alpha, \beta\}\)-colored cycles). If \( M \) is a closed surface, let \( y_1 \) denote the unique \( \{\alpha, \beta\}\)-colored cycle in \( G \). If the dimension of \( M \) is greater than 2, let \( Y = \{y_1, \ldots, y_q\} \) be the set of all \( \{\alpha, \beta\}\)-colored cycles, but one, arbitrarily chosen. Fix an orientation and a starting point for each of them. Compose the word \( r_j \) from the cycle \( y_j \) by the following rule. Follow the chosen direction, starting from the chosen vertex and list consecutively every \( x_i \) you meet with exponent +1 or −1 according to whether \( \alpha \) or \( \beta \) is the color of the edge which leads you into \( x_i \). If \( R = \{r_1, \ldots, r_q\} \), then \( \langle X | R \rangle \) is a finite presentation for \( \pi_1(M) \).

**Algorithm 2** [51][52]. Let \( H \) be a subgraph of \( G \). An edge \( e \in E(G) \setminus E(H) \) is said to be dependent on \( H \) if there is a \( k \)-colored cycle, \( k < n+1 \), containing \( e \), all of whose other edges lie in \( H \). Let \( H = H_0, \ldots, H_m = H^* \) be a sequence of subgraphs of \( G \) such that \( H_{i+1} = H_i + e \), where \( e \) is dependent on \( H_i \), and there is no edge in \( G \) dependent on \( H^* \). Then \( H^* \) is called the closure of \( H \) in \( G \). Construct a spanning tree \( T \) in \( G \). Arbitrarily assign an orientation
each edge in \( X = E(G) \setminus E(T^*) \), where \( T^* \) is the closure of \( T \) in \( G \). For every \( 2 \)-colored cycle \( \sigma \) of \( G \), let \( r_\sigma \) be the sequence of edges in \( X \) around \( \sigma \), each with exponent +1 or −1 according to the orientation. If \( R \) is the set of words \( r_\sigma \), then \( \langle X|R \rangle \) is a finite presentation for \( \pi_1(M) \).

\[ \text{Algorithm 3} \ [38][39]. \text{ The set of generators is the vertex set of } G, \text{ i.e., } X = V(G). \text{ Choose two colors } \alpha \text{ and } \beta \text{ in } \Delta_n, \text{ and let } C \text{ be the set of all } \{ \alpha, \beta \} \text{–colored cycles of } G. \text{ For every } \sigma \in C \text{ of length } 2m, \text{ let } r_\sigma \text{ be the word } x_1x_2^{-1} \cdots x_{2m-1}x_{2m}^{-1}, \text{ where } x_i \in X \text{ is the } i \text{–vertex of } \sigma. \text{ The initial vertex } x_1 \text{ and the direction around } \sigma \text{ can be chosen arbitrarily, except that the edge with ends } x_1 \text{ and } x_2 \text{ should be colored by } \alpha. \text{ Let } R_1 \text{ denote the set of words } r_\sigma \text{ for every } \sigma \in C. \text{ Let } R_2 \text{ denote the set of words } xy^{-1}, \text{ where } x, y \in X \text{ and } x, y \text{ belong to the same connected component of the partial } (n-1) \text{–subgraph obtained by deleting the colors } \alpha \text{ and } \beta. \text{ If } R = R_1 \cup R_2 \cup \{x_0\}, \text{ where } x_0 \text{ is an arbitrary vertex of } G, \text{ then } \langle X|R \rangle \text{ is a finite presentation for } \pi_1(M). \]

\[ \text{Theorem 4.1} \text{ Let } M \text{ be a closed connected (PL) } n \text{–manifold, } (G, c) \text{ a crystallizations of } M, \text{ and } \langle X|R \rangle \text{ a finite presentation arising from } G \text{ by one of the algorithms discussed above. Then } \langle X|R \rangle \text{ defines a group which is isomorphic to } \pi_1(M). \]

\[ \text{Proof.} \ 1) \text{ Let } K = K(G) \text{ be the contracted triangulation of } M, \text{ and } \{v_i; i \in \Delta_n\} \text{ the vertex–set of } K. \text{ We may assume that } v_i \text{ corresponds to } G_i, i \in \Delta_n, \text{ and that it is colored by } i. \text{ For any pair of distinct colors } \alpha \text{ and } \beta \text{ in } \Delta_n, \text{ let } K(\alpha, \beta) \text{ be the one–dimensional subcomplex of } K \text{ generated by the vertices } v_\alpha \text{ and } v_\beta, \text{ and } K(\hat{\alpha}, \hat{\beta}) \text{ the } (n-2) \text{–subcomplex generated by the vertices colored by } \Delta_n \setminus \{\alpha, \beta\}. \text{ If } SdK \text{ is the first barycentric subdivision of } K, \text{ let } H(\alpha, \beta) \text{ be the largest subcomplex of } SdK \text{ disjoint from } SdK(\alpha, \beta) \cup SdK(\hat{\alpha}, \hat{\beta}). \text{ Then the polyhedron underlying } H(\alpha, \beta) \text{ is a closed } (n-1) \text{–manifold which splits } M \text{ into two complementary bordered } n \text{–manifolds } N(\alpha, \beta) \text{ and } N(\hat{\alpha}, \hat{\beta}). \text{ Furthermore, } N(\alpha, \beta) \text{ and } N(\hat{\alpha}, \hat{\beta}) \text{ are regular neighborhoods in } M \text{ of the polyhedrons underlying } K(\alpha, \beta) \text{ and } K(\hat{\alpha}, \hat{\beta}), \text{ respectively. In particular, } N(\alpha, \beta) \text{ is an } n \text{–dimensional handlebody. Since the edges of } K(\alpha, \beta), \text{ but one, correspond bijectively to the generators in } X \text{ from Algorithm 1, } \pi_1(N(\alpha, \beta)) \text{ is the free group on the set } X. \text{ For every } (n-2) \text{–simplex } \sigma \in K(\hat{\alpha}, \hat{\beta}), \text{ but one arbitrarily chosen, let } J_\sigma \text{ be the simple circle in } \partial N(\alpha, \beta) \text{ which is isotopic to the disjoined link of } \sigma \text{ in } K. \text{ Attach a copy of } B^2 \times I^{n-2} \text{ (} I = [0, 1] \text{)} \text{ to } N(\alpha, \beta) \text{ by identifying } \partial B^2 \times I^{n-2} \text{ with a regular neighborhood of } J_\sigma \text{ in } \partial N(\alpha, \beta). \text{ The resulting manifold } M_1 \text{ has an } (n-1) \text{–sphere boundary, and } M \text{ can be obtained from } M_1 \text{ by attaching a copy of a } (n-1) \text{–sphere boundary to each } J_\sigma \text{ on } \partial N(\alpha, \beta). \]
by attaching an $n$–cell (in particular, $\pi_1(M_1) \cong \pi_1(M)$). Let $r_\sigma$ denote a word in the generators of $X$ representing an element of $\pi_1(N(\alpha, \beta))$ determined by $J_\sigma$. Then the set of words $r_\sigma$ is precisely $R$ from Algorithm 1.

2) Each based loop in $\pi_1(K)$ is represented by a based simple cycle in $G$, which is the dual 1–skeleton of $K$. An edge $e \in E(G) \setminus E(T)$ represents a unique single circle in $T + e$, while an edge in $T^*$ is null homotopic in $M$. Hence $X$ from Algorithm 2 is a set of generators for $\pi_1(K)$. Every 2–colored cycle in $G$ corresponds to a null homotopic loop in $M$, hence the relation set $R$ from Algorithm 2.

3) Algorithm 3 is dual to Algorithm 2. Here loops in $M$ are represented by based simple edge cycles in $K$ rather than $G$. Hence $\pi_1(K)$ is isomorphic to the standard edge–path group of $K$ whose elements are closed simple edge paths in the 1–skeleton of $K$, and closed edge paths around 2–simplexes are null homotopic in $M$. To see the equivalence between Algorithms 1 and 3, it suffices to note that relations from $R_2$ identify all the vertices of a given connected component of the partial $(n - 1)$–subgraph $G_{\Delta \setminus \{\alpha, \beta\}}$. Finally, relations of $R_1$ one-to-one correspond to those in $R$ from Algorithm 1. □

Problem 4.1 Deduce combinatorial descriptions of the homotopy groups $\pi_i(M), i \geq 2$, from a crystallization of $M$.

Problem 4.2 Under which combinatorial conditions on a crystallization of a manifold $M$ certain homotopy groups of $M$ vanish?

Combinatorial analogues to simple–homotopy for cell complexes were developed for graphs by several authors (see for example [2],[27], and [43]).

Problem 4.3 Study relations among the various homotopy theories for graphs, and compare the above algorithms with those described in the quoted papers.

5. Homology

A homology theory for colored graphs was developed in [21] and [22]. Let $(G, c_G)$ be an $(n + 1)$–colored graph. Two vertices $v$ and $w$ of $G$ are said to be $\Gamma$–connected, $\Gamma \subset \Delta_n$, if they are joined by a finite sequence of edges with colors in $\Gamma$. Let us define

$$S_k(G) = \{(\Gamma, V): \# \Gamma = k \land V \text{ is a } \Gamma\text{–connected component of } G\}.$$

The free $R$–module ($R$ a principal ideal domain) $C_k(G)$ on the set $S_k(G)$ is called the module of $k$–chains of $G$. For simplicity, we shall suppress the $R$–coefficients. We set $C_*(G) = \{C_k(G)\}_{k \in \mathbb{Z}}$ and $C_k(G) = 0$ if $S_k(G) = \emptyset$. The
boundary operator $\partial$ on $C_*(G)$ is defined as follows. If $A \subset \mathbb{Z}$ is a finite nonempty set, let $\sigma_A : A \to A$ be the cyclic permutation defined by

$$\sigma_A(m) = \begin{cases} \min \{z : z \in A \land z > m\} & \text{if } m < \max A \\ \min A & \text{if } m = \max A \end{cases}$$

Let $\text{ord}(A, m)$ denote the integer such that $\sigma_A^{\text{ord}(A, m)}(\min A) = m$. Then we define $\partial : C_*(G) \to G_*(G)$ by setting

$$\partial(\Gamma, V) = \sum_{i \in \Gamma} (-1)^{\text{ord}(\Gamma, i)} \sum \{(\Gamma \setminus \{i\}, W)\}$$

where the second summation is taken over all pairs $(\Gamma \setminus \{i\}, W)$ such that $W$ is a $(\Gamma \setminus \{i\})$-connected component of $G$ with $W \subset V$. It was proved in [21] that $\partial C_{p+1}(G) \subset C_p(G)$ and $\partial \circ \partial = 0$, hence the pair $(C_*(G), \partial)$ is a chain complex. Therefore, the homology $H_*(C_*(G))$ and the cohomology $H^*(C_*(G))$ are defined in the usual way, and denoted by $H_*(G)$ and $H^*(G)$, respectively. Any colored morphism (resp. isomorphism) between colored graphs induces homomorphisms (resp. isomorphisms) between the respective (co)homology modules, and the correspondences are functorial. The following result describes the topological meaning of the constructed (co)homology of colored graphs.

**Theorem 5.1** Let $(G, c_G)$ be an $(n+1)$–colored graph, and $K = K(G)$ the $n$–dimensional pseudocomplex represented by $G$. Then we have $H_p(G) \cong H^{n-p}(|K|)$ and $H^p(G) \cong H_{n-p}(|K|)$. If $G$ represents an orientable closed homology $n$–manifold, then $H_p(G) \cong H^{n-p}(G)$, $0 \leq p < n$.

For combinatorial analogs of exact homology sequences, products, duality, etc. see [21]. The computation of the integral homology groups $H_*(G)$ can be performed by integer matrices arising from the $(n+1)$–colored graph $(G, c_G)$. For any $k$, $0 \leq k < n$, we consider the integer matrix $E^{(k)} = (\eta_{ij}^{(k)})$ which describes the incidence relation between the $k$– and $(k+1)$–colored components of $G$. The elements of $E^{(k)}$ are defined as follows. Given $(\Gamma_i, V_i) \in S_k(G)$ and $(\Theta_j, W_j) \in S_{k+1}(G)$, we put $\eta_{ij}^{(k)} = 0$ when $(\Gamma_i, V_i)$ and $(\Theta_j, W_j)$ are not incident, i.e., either $\Gamma_i \not\subset \Theta_j$ or $\Gamma_i \subset \Theta_j$ and $V_i \not\subset W_j$. If they are incident, i.e., $\Gamma_i \subset \Theta_j \subset \Delta_n$ and $V_i \subset W_j$, let $(\epsilon_0, \epsilon_1, \ldots, \epsilon_k)$ be the permutation of $\Theta_j$ induced by the fundamental cycle $(0, 1, \ldots, n)$ of $\Delta_n$. Then we put $\eta_{ij}^{(k)} = (-1)^h$, where $\Gamma_i = \Theta_j \setminus \{\epsilon_h\}$, $0 \leq h \leq k$. The matrices $E = \{E^{(k)}\}_{k \geq 0}$, called the incidence matrices of the $(n+1)$–colored graph $(G, c_G)$, completely
determine the graph and its underlying polyhedron. Let $F^{(k)} = (\mu_{ij}^{(k)})$ be the matrix obtained from $E^{(k)}$ (by integer unimodular transformations) such that 

$\mu_{ij}^{(k)} = 0$, $i \neq j$, $r_i^{(k)} = \mu_{ii}^{(k)} > 0$, $1 \leq i \leq \nu_k$, $\nu_k = \text{rank} E^{(k)}$ and $r_{i-1}^{(k)}$ is an integer multiple of $r_i^{(k)}$. Then the $k$th Betti number of $K(G)$ is given by 

$\beta_k = \alpha_k - \nu_k - \nu_k - 1$ where $k \in \Delta_n$, $\nu_{-1} = \nu_n = 0$, and $\alpha_k$ is the number of all $k$–colored components of $G$. Suppose that there is an integer $\rho_k$ (possibly zero) such that $r_i^{(k)} > 1$ for every $i$, $1 \leq i \leq \rho_k$, and $r_i^{(k)} = 1$ for every $i$, $\rho_k < i \leq \nu_k$. Then $H_k(G)$ is isomorphic to $\oplus \beta_k \mathbb{Z} \oplus \mathbb{Z}_{r_1^{(k)}} \oplus \cdots \oplus \mathbb{Z}_{r_{\rho_k}^{(k)}}$. Further applications and several examples of computation can be found in [22].

**Problem 5.1** Find combinatorial conditions to characterize crystallizations of homology spheres in dimension $n \geq 3$.

6. Regular genus

The representation of combinatorial manifolds via crystallizations allows to extend in higher dimension the classical concept of genus for 2– and 3–manifolds. A 2–cell embedding of an $(n+1)$–colored graph $(G, c_G)$ on a closed connected surface $F$ is called regular if there is a cyclic permutation $\epsilon = (\epsilon_0, \ldots, \epsilon_n)$ of the color set $\Delta_n$ such that each 2–cell of $F \setminus |G|$ is bounded by a cycle of $G$ with edges alternatively colored by $\epsilon_i$ and $\epsilon_{i+1}$ (indices mod $n+1$). The **regular genus** $g(G)$ of $(G, c_G)$ is the smallest integer $h$ such that $(G, c_G)$ regularly embeds into a closed surface of genus $h$. It is easily seen that the surface into which $G$ embeds is orientable if and only if $G$ is bipartite. For a closed connected (PL) $n$–manifold $M$, the **regular genus** $g(M)$ of $M$ was defined in [34] as the nonnegative integer

$$g(M) = \min \{g(G) : (G, c_G) \text{ is a crystallization of } M\}.$$ 

It is proved in [34] that the regular genus is a (PL) manifold invariant which extends to arbitrary dimension the classical genus of a closed surface and the Heegaard genus of a closed 3–manifold. This invariant permits to characterize spheres and handles of arbitrary dimension, and it is an upper bound for the rank of the fundamental group of a manifold.

**Theorem 6.1**

i) [30] The only genus zero $n$–manifold, $n \geq 2$, is the standard $n$–sphere.

ii) [13][19][24] Let $M^n$ be a closed connected $n$–manifold, $n \geq 4$. Then $g(M) = 1$ if and only if $M$ is (PL) homeomorphic to $S^1 \times S^{n-1}$. 

iii) [4] For every $n$–manifold $M^n$, $n \geq 3$, $g(M^n) \geq rk(M^n)$, where $rk(M^n)$ denotes the rank of the fundamental group $\pi_1(M^n)$.

These characterizations suggest to study the relations between the regular genus and the topology of manifolds. There are several results especially in dimensions 4 and 5.

**Theorem 6.2** Let $M^4$ be a smooth (or PL) closed connected 4–manifold of genus $g = g(M)$.

1) [14] If $g = 2$ and $M$ is orientable, then $M$ is (PL) homeomorphic to either the projective complex plane $\mathbb{C}P^2$ or the connected sum $\#_2(S^1 \times S^3)$. If $g = 2$ and $M$ is nonorientable, then $M$ is (PL) homeomorphic to $S^1 \times S^3$ (the twisted $S^3$–bundle over $S^1$).

2) [20] If $g = 3$, then $M$ is (PL) homeomorphic to either $\#_3(S^1 \times S^3)$ or $\mathbb{C}P^2 \#(S^1 \times S^3)$.

3) [14] If $g = 4$ and $M$ is nonorientable, then $M$ is (PL) homeomorphic to either $\#_4(S^1 \times S^3) \sim (S^1 \times S^3)$ or $(S^1 \times S^3) \#(S^1 \times S^3)$.

4) [20] If $g = 4$ and $M$ is orientable, then $M$ is (TOP) homeomorphic to one of the following manifolds: $\#_4(S^1 \times S^3)$, $\#_2(S^1 \times S^3) \# \mathbb{C}P^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$, $S^2 \times S^2$ and $S^2 \times S^2$ (the twisted $S^2$–bundle over $S^2$).

5) [49] If $g = 5$, then $M$ is (TOP) homeomorphic to one of the following manifolds: $\#_5(S^1 \times S^3)$, $\#_3(S^1 \times S^3) \# \mathbb{C}P^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2 \#(S^1 \times S^3)$, $(S^2 \times S^2) \#(S^1 \times S^3)$ and $(S^2 \times S^2) \#(S^1 \times S^3)$.

6) [49] If $g = 6$ and $M$ is prime and orientable, then $M$ is (TOP) homeomorphic to a lens–fiber bundle over the circle. If further $M$ is spin, then $M$ is $L(p, q) \times S^1$, $q \neq 0$, possibly including the case $L(0, 1) = S^1 \times S^2$.

7) [14] [20] If $g = 6$ and $M$ is nonorientable, then $M$ is (TOP) homeomorphic to either $(\pm \mathbb{C}P^2) \#(S^1 \times S^3)$, $\mathbb{R}P^4$ or $\#_3(S^1 \otimes S^3)$, where $S^1 \otimes S^3$ represents either $S^1 \times S^3 \sim S^1 \times S^3$.

This result, part 4) (resp. part 7)) characterizes $S^2 \times S^2$ (resp. $\mathbb{R}P^4$) among closed orientable (resp. nonorientable) 4–manifolds as the unique prime smooth (or PL) closed orientable (resp. nonorientable) 4–manifold of genus 4.
(resp. 6), up to (TOP) homeomorphism. Since $g(\mathbb{R}P^3 \times S^1) = 6$ (see Theorem 6.10f in this section), we conjecture that the genus six characterizes $\mathbb{R}P^3 \times S^1$ among closed connected prime orientable 4–manifolds. The next result follows from some relations obtained in [13], p.1011, and [14], p.205 (see also [20], p.38, and [35]).

**Theorem 6.3** Let $M^4$ be a closed connected smooth (or PL) 4–manifold of genus $g = g(M)$, and $(G,c_G)$ a crystallization of $M$. Then we have:

$$2 - 2g \leq \chi(M) = 2 - 2g + \sum_{i=0}^{4} g_i \leq 2 + \left\lfloor g/2 \right\rfloor$$

$$rk(M) \leq \min\{g - g_i - g_j : i, j, \in \Delta_4, \ i \neq j\}$$

where $g_i$ is the regular genus of the partial subgraph $G_i$, $i \in \Delta_4$, $\chi(M)$ is the Euler characteristic of $M$ and $\left\lfloor x \right\rfloor$ is the integer part of $x$.

**Corollary 6.4** [14] Let $M^4$ be a closed connected smooth (or PL) orientable 4–manifold of genus $g$. Then

$$b_2(M) \leq \left\lfloor \frac{5}{2}g \right\rfloor$$

If $M$ is simply-connected, then

$$b_2(M) \leq \left\lfloor \frac{g}{2} \right\rfloor$$

where $b_k(M)$ denotes the $k$th Betti number of $M$.

**Corollary 6.5** [14]

1. $g(\#_k(S^1 \times S^3)) = k$
2. $g(\#_k(\pm \mathbb{C}P^2)) = 2k$
3. $g(\#_k(S^2 \times S^2)) = 4k$

From Corollary 6.4 and the Freedman classification of simply-connected (TOP) 4–manifolds, we get the following result (see [14], p.205)

**Theorem 6.6** If $M^4$ is a closed simply-connected smooth (or PL) 4–manifold of genus $g \leq 31$, then $M$ is (TOP) homeomorphic to either $\#_h(\pm \mathbb{C}P^2)$ or $\#_h(S^2 \times S^2)$, where $h = b_2(M)$.

The following theorem was proved in [20], Theorem 1, p.40.
Theorem 6.7 Let $M^4$ be a smooth (or PL) closed orientable connected 4–manifold of genus $g = g(M)$, and $(G, c_G)$ a crystallization of $M$. Set $\Delta = \sum_{i=0}^4 g_i$. If $\Delta = 0$, then $M$ is (PL) homeomorphic to the connected sum $\#_g(S^1 \times S^3)$. If $\Delta = 5$, then $M$ is (PL) homeomorphic to the connected sum $\#_{g-2}(S^1 \times S^3) \# \mathbb{C}P^2$. There are no $M^4$ with $1 \leq \Delta \leq 9$ and $\Delta \neq 5$.

As a consequence of Theorem 6.7 we give a simple proof of the following theorem which improves the main results of [6] and [11] (see also [3], p.300, and [7], p.68).

Theorem 6.8 Let $M^4$ be a smooth (or PL) closed orientable connected 4–manifold. Then $g = g(M) = \rho k(M)$ if and only if $M$ is (PL) homeomorphic to $\#_g(S^1 \times S^3)$. Further, $g = g(M) = \rho k(M) + 2$ if and only if $M$ is (PL) homeomorphic to $\#_{g-2}(S^1 \times S^3) \# \mathbb{C}P^2$. No $M^4$ exists with $g(M^4) = \rho k(M) + \rho$, $\rho \in \{1, 3\}$. No $M^4$ exists with regular genus $g$ and $3 - 2g \leq \chi(M) \leq 11 - 2g$, $\chi(M) \neq 7 - 2g$.

Proof. If $\rho k(M^4) = g(M^4) = g$, by Theorem 6.3 $g \leq g_i - g_j$, $i, j \in \Delta_4$, $i \neq j$, hence $0 \leq g_i + g_j \leq 0$, i.e., $g_i = 0$ for every $i \in \Delta_4$. Then $\Delta = 0$, and the result follows from Theorem 6.7. If $\rho k(M^4) = g - 2$, by Theorem 6.3 we get $0 \leq g_i + g_j \leq 2$. Suppose for example $g_0 = 2$. Then $g_i = 0$ for $i = 1, 2, 3, 4$ and $\Delta = 2$ which contradicts Theorem 6.7. Thus $g_i = 1$ for every $i \in \Delta_4$, and whence $\Delta = 5$. Now we apply Theorem 6.7. If $\rho k(M^4) = g - 1$, by Theorem 6.3 we get $0 \leq g_i + g_j \leq 1$. Then we have $\Delta \leq 3$ and the result follows from Theorem 6.7. If $\rho k(M^4) = g - 3$, by Theorem 6.3 we get $0 \leq g_i + g_j \leq 3$. Then we have $\Delta \leq 9$. Suppose $\Delta = 5$. Theorem 6.7 implies $M \cong_{PL} \#_{g-2}(S^1 \times S^3) \# \mathbb{C}P^2$, where $g = g(M^4)$ and $\rho k(M^4) = g - 2$; but this contradicts the hypothesis. The last statement follows from Theorems 6.3 and 6.7. \hfill $\Box$

Problem 6.1 Classify all closed connected orientable 4–manifolds $M^4$ such that $g(M) = \rho k(M) + \rho$, $\rho \geq 4$.

Problem 6.2 Classify all closed connected orientable 4–manifolds $M^4$ such that $\chi(M) = 2 - 2g + \delta$, $\delta \geq 10$.

Examples:

$M = S^2 \times S^2$, $g(M) = 4$, $\rho = 4$, $\delta = 10$

$M = \mathbb{R}P^3 \times S^1$, $g(M) = 6$, $\rho = 4$, $\delta = 10$

$M = \mathbb{C}P^2 \# \mathbb{C}P^2$, $g(M) = 4$, $\rho = 4$, $\delta = 10$

We conjecture that the number of prime manifolds satisfying conditions in the above problems is finite.
We now discuss some relations between the regular genus and the 4–dimensional Poincaré Conjecture, denoted $P(4)$ (see [14], Section 5, p.212).

**Conjecture C(1).** If $M$ is a closed smooth (or PL) simply-connected 4–manifold, then $g(M) = 2b_2(M)$.

**Conjecture C(2).** The regular genus is additive with respect to the connected sum of simply-connected smooth (or PL) 4–manifolds.

We prove that $C(1)$ is equivalent to $C(2)$.

$C(1) \Rightarrow C(2)$. If $M$ and $M'$ are closed smooth (or PL) simply-connected 4–manifolds, then we have

\[ g(M \# M') = 2b_2(M \# M') = 2b_2(M) + 2b_2(M') = g(M) + g(M'). \]

$C(2) \Rightarrow C(1)$. If $M$ is a closed smooth (or PL) simply-connected 4–manifold, then there exist two integers $h$ and $k$ such that $M \# hC^2 \# k(-C^2)$ is diffeomorphic to $aC^2 \# b(-C^2)$, where $a = h + (b_2(M) + \sigma(M))/2$, $b = k + (b_2(M) - \sigma(M))/2$ and $\sigma(M)$ is the signature of $M$. (Here $hC^2$ denotes the connected sum of $h$ copies of $C^2$). Then we have

\[
g(M \# h(C^2) \# k(-C^2)) = g(aC^2 \# b(-C^2)) = 2(a + b) = 2(h + k) + 2b_2(M) = g(M) + g(h(C^2) \# k(-C^2)) = g(M) + 2(h + k)
\]

hence $g(M) = 2b_2(M)$.

Obviously $C(1)$ (or $C(2)$) implies $P(4)$ as follows. Let $M$ be a closed smooth (or PL) homotopy 4–sphere. Since $b_2(M) = 0$, we have $g(M) = 2b_2(M) = 0$, hence $M$ is the genuine 4–sphere.

**Theorem 6.9** [14] If $M^4$ is a smooth homotopy 4–sphere, then there exists an integer $k$ such that

\[ g(M \# k(S^2 \times S^2)) = 4k. \]

This suggests a further conjecture.

**Conjecture C(3).** If $M$ is a smooth (or PL) homotopy 4–sphere, then

\[ g(M \# k(S^2 \times S^2)) \geq g(M) + 4k \]
for every integer $k$.

We prove that $P(4)$ and $C(3)$ are equivalent. If $M$ is a homotopy 4–sphere, then there exists an integer $k$ such that

$$4k = g(M \# k(S^2 \times S^2)) = g(k(S^2 \times S^2)) \geq g(M) + 4k$$

hence $g(M) = 0$, and $M$ is a genuine 4–sphere. Conversely, if $M$ is a smooth homotopy 4–sphere, then $M$ is (PL) homeomorphic to $S^4$. Then we have

$$g(M \# k(S^2 \times S^2)) = g(k(S^2 \times S^2)) = 4k \geq g(M) + 4k$$

since $g(M) = 0$.

A lot of estimations are made on the regular genus of some closed connected $n$–manifolds; here $T_h$ and $U_k$ denote the orientable surface of genus $h$ and the nonorientable surface of genus $k$.

**Theorem 6.10**

a) [31] [35]

$$g(T_h \times S^1) = 2h + 1.$$  
$$g(U_k \times S^1) = 2k + 2$$

b) [31]

$$2(h + h') \leq g(T_h \times T_{h'}) \leq 2(6hh' + 3(h + h') + 2)$$
$$2(2h + k) \leq g(T_h \times U_k) \leq 2(6hk + 3(2h + k) + 4)$$
$$2(k + k') \leq g(U_k \times U_{k'}) \leq 2(3kk' + k + k') + 4$$

c) [35]

$$2h \leq g(T_h \times S^2) \leq 6h + 4$$
$$k \leq g(U_k \times S^2) \leq 3k + 4$$

d) [35]

$$2h \leq g(T_h \times S^3) \leq 8(2h + 1)$$
$$k \leq g(U_k \times S^3) \leq 8(k + 1)$$
e) [28] For each $n \geq 3$:
\[
g(S^n \times S^2) \leq n^2 - 1
\]
\[
g(S^n \times S^3) \leq \frac{2}{3} n^3 + n^2 - \frac{2}{3} n
\]
It was conjectured in [28], p.420, that, for each $n \geq 3$, $g(S^n \times S^2) = n^2 - 1$. This is true for $n = 3$, but the conjecture is false for $n = 4$ and $n = 5$, as shown recently in [50].

f) [25] For each $n \geq 2$,
\[
g(\mathbb{R}P^n) = \begin{cases} 
2^{n-3}(n - 3) + 1 & \text{if } n \text{ is odd} \\
2^{n-2}(n - 3) + 2 & \text{if } n \text{ is even.}
\end{cases}
\]

g) [49] If $L(p,q)$ is the lens space of type $(p,q)$, $p > q \geq 1$, then
\[
6 \leq g(L(p,q)) \times S^1 \leq \min \{6p - 6, 4p + 1\}.
\]
In particular, the regular genus of $\mathbb{R}P^3 \times S^1$ is exactly 6.

h) [14] If $V_n$ is an algebraic non singular hypersurface of degree $n$ in $\mathbb{C}P^3$, then we have
\[
[g/2] \geq n^3 - 4n^2 + 6n - 2
\]

i) [14] If $V$ is a simply-connected complex surface, then
\[
[g/2] \geq 12p_{\text{geom}} + 10 - c_1^2[V].
\]
If $V$ is minimal elliptic, then
\[
[g/2] \geq 12p_{\text{geom}} + 10.
\]
Here $p_{\text{geom}}$ and $c_1[V]$ denote the geometric genus and the first Chern class of $V$, respectively.

In dimension 5 we have the following results.

**Theorem 6.11** Let $M^5$ be a smooth (or PL) closed orientable connected 5–manifold of genus $g = g(M^5)$. Then we have:

i) [9] [24] $M$ is (PL) homeomorphic to $\#_g(S^1 \times S^4)$ if and only if $g(M) = \text{rk}(M)$. 


ii) [10] If \( g(M^5) \leq 8 \), then \( M \) is (PL) homeomorphic to either \( \#_g(S^1 \times S^4) \), \( S^2 \times S^3 \) or \( S^2 \times S^3 \). In particular, \( g(S^2 \times S^3) = g(S^2 \times S^3) = 8 \).

iii) [9] If \( \pi_1(M^5) \cong \ast_p \mathbb{Z} \) and \( g(M) = p + 8 \), then \( M \) is (PL) homeomorphic to either \( \#_p(S^1 \times S^4) \#(S^2 \times S^3) \) or \( \#_p(S^1 \times S^4) \#(S^2 \times S^3) \).

To end the section we state some further conjectures.

**Conjecture C(4).** [30] Let \( M \) and \( M' \) be closed connected orientable PL \( n \)–manifolds. Then

\[
g(M \# M') = g(M) + g(M')
\]

Theorem 6.11, part (ii) together with the classification of simply-connected 5–manifolds [1][48] and the Perelman solution of the 3–dimensional Poincaré Conjecture gives the following result

**Theorem 6.12** Let \( M^3 \) be a closed connected orientable 3–manifold. Then \( g(M^3 \times S^2) = 8 \) if and only if \( M^3 \) is a genuine 3–sphere.

This suggests the following conjectures (see [10] and [3], p.303, for \( n = 3 \) and \( m = 2 \)):

**Conjecture C(5).** For every \( n \geq 3 \) and \( m \geq 2 \),

\[
g(M^n \times S^m) = g(S^n \times S^m) \Leftrightarrow M^n \cong_{PL} S^n
\]

A closed connected PL 4–manifold \( M \) is said to be a \( C(p, q) \)–manifold if \( M = B_0 \cup B_1 \cup B_2 \), where \( B_i, i = 0, 1, 2 \), is a closed 4–ball and \( B_i \cap B_j = \partial B_i \cap \partial B_j \) is a (possibly non–connected) 3–manifold for every \( i \neq j \). This class of manifolds was introduced by H. Ikeda and M. Yamashita in [37], and successively studied in [18]. The role of the integers \( p \) and \( q \) is explained as follows. If \( M \in C(p, q) \), then \( \pi_1(M) \) is a free group of rank \( p \), and the second integral homology of \( M \) is the abelian free group of rank \( q \). Any simply–connected closed smooth 4–manifold which admits a special handlebody decomposition without 1– and 3–handles belongs to \( C(0, q) \) for some \( q \). In particular, every nonsingular hypersurface \( V_n \) of degree \( n \) in \( \mathbb{C}P^3 \) is a \( C(0, q) \)–manifold, where \( q = n^3 - 4n^2 + 6n - 2 \). Recall that \( V_1 = \mathbb{C}P^2 \), \( V_2 = S^2 \times S^2 \), \( V_3 = \mathbb{C}P^2 \# 6 \mathbb{C}P^2 \) and \( V_4 = K_3 \) (the Kummer surface). It is an
open question whether any simply–connected smooth 4–manifold belongs to $C(0, q)$ for some $q$.

**Conjecture C(6).** The regular genus of a simply–connected $M \in C(p, q)$ is $2q$. In particular, $g(V_n) = 2(n^3 - 4n^2 + 6n - 2)$, for every $n \geq 1$.

This is true for $V_n$, $n \leq 3$.

Let $X$ be a smooth closed connected spin 4–manifold, $b_2(X)$ the second Betti number, $\sigma(X)$ the signature, and $g(X)$ the regular genus of $X$. In [44], Y. Matsumoto conjectured the following inequality $b_2(X) \geq \frac{11}{8} |\sigma(X)|$ which has been called the $\frac{11}{8}$–conjecture in the current literature. It is known that all complex surfaces and their connected sums satisfy the conjecture.

**Conjecture C(7).** If $X$ is a smooth closed connected spin 4–manifold, then

$$\left[ \frac{5}{2} g(X) \right] \geq \frac{11}{8} |\sigma(X)|.$$ 

If $X$ is simply–connected, then

$$g(X) \geq \frac{11}{4} |\sigma(X)|.$$ 

### 7. Complexity

Let $M^n$ be a closed connected orientable (PL) $n$–manifold, and $(G, c_G)$ a crystallization of $M$. Let $\beta_k(G) = \text{rank} C_{n-k}(G)$ denote the number of $k$–simplexes of $K(G)$. The $k$–complexity of $M$, $0 \leq k \leq n$, was defined in [4] as

$$c_k(M) = \min \{ \beta_k(G) : (G, c_G) \text{ is a crystallization of } M \}.$$ 

Since one always has at least $\binom{n+1}{k+1}$ $k$–simplexes in $K(G)$, it will be useful to define the reduced $k$–complexity as

$$\tilde{c}_k(M) = c_k(M) - \binom{n+1}{k+1} \quad 0 \leq k \leq n - 1$$

$$\tilde{c}_n(M) = c_n(M) - 2$$

The case $k = n$ is special because any regular graph has at least two vertices. Of course, $\tilde{c}_n(M) = 0$ if and only if $M \cong S^n$. Another way to write Pezzana’s
A. Cavicchioli and F. Spaggiari

Theorem 7.1 [21]

i) For any closed connected surface $M$, $\tilde{c}_1(M) = \frac{3}{2} \tilde{c}_2(M) = 6 - 3\chi(M)$. In particular, the reduced complexities are additive in dimension two.

ii) If $M$ is a closed connected 3–manifold, then $\tilde{c}_1(M) = \frac{1}{2} \tilde{c}_2(M) = \tilde{c}_3(M)$.

iii) If $M$ is a closed connected 4–manifold, then $\tilde{c}_1(M) = 6 - 3\chi(M) + \frac{1}{2} \tilde{c}_4(M)$, $\tilde{c}_2(M) = 4 - 2\chi(M) + 2\tilde{c}_4(M)$ and $\tilde{c}_3(M) = \frac{5}{2} \tilde{c}_4(M)$.

iv) If $M$ is a homology 4–sphere, then $\tilde{c}_1(M) = 2\tilde{c}_1(M) = \frac{1}{2} \tilde{c}_2(M) = \frac{2}{5} \tilde{c}_3(M)$.

By iii), the additivity of $\tilde{c}_4$ in dimension four implies that $\tilde{c}_k$ is additive too for $k \leq 3$.

Conjecture $C(8)$. Let $M$ and $M'$ be two closed orientable PL 4–manifolds. Then

$$\tilde{c}_4(M \# M') = \tilde{c}_4(M) + \tilde{c}_4(M').$$

Conjecture $C(8)$ implies the (PL) 4–dimensional Poincaré Conjecture $P(4)$. For this, let $M$ be a homotopy 4–sphere. Then there exists a nonnegative integer $h$ such that $M \# h(S^2 \times S^2)$ is diffeomorphic to $h(S^2 \times S^2)$. Now $C(8)$ gives $\tilde{c}_4(M \# h(S^2 \times S^2)) = \tilde{c}_4(M) + \tilde{c}_4(h(S^2 \times S^2)) = \tilde{c}_4(h(S^2 \times S^2))$, hence $\tilde{c}_4(M) = 0$. Thus we have $M \cong S^4$.

Since $\tilde{c}_4(h(S^2 \times S^2)) = 12h$, $C(8)$ can be reduced to the following

Conjecture $C(9)$. For every closed orientable 4–manifold $M$,

$$\tilde{c}_4(M \# h(S^2 \times S^2)) = \tilde{c}_4(M) + 12h.$$

The next conjectures are equivalent to $P(4)$.

Conjecture $C(10)$. If $M^4$ is a simply-connected closed PL 4–manifold, then

$$\tilde{c}_4(M) = 6\chi(M) - 12 = -12g(M) + 6\Delta$$

or, equivalently, $\tilde{c}_1(M) = 0$.

Conjecture $C(11)$. If $M^4$ is a homotopy 4–sphere, then $\tilde{c}_1(M) = 0$. 
The notion of complexity gives rise to an interesting family of crystallizations (see [4], Section 3, p.316). A crystallization \((G, c_G)\) of a closed connected orientable (PL) \(n\)-manifold \(M^n\) is said to be minimal if \(c_n(M) = \beta_n(G)\).

**Problem 7.1** Give combinatorial characterizations of minimal crystallizations, and study their properties.

The complete classification of all closed connected prime orientable 3–manifolds \(M\) with reduced complexity \(\tilde{c}_3(M) \leq 26\) was obtained in [40], Section 5. There are exactly sixty-nine of such manifolds. Among them, there are \(S^3\), \(S^2 \times S^1\), twenty-eight lens spaces, the six Euclidean orientable 3–manifolds, and sixteen quotients of \(S^3\) by the action of their finite (non-cyclic) fundamental groups. The following is the main theorem of [26].

**Theorem 7.2**

(a) There are no closed connected 4–manifolds \(M\) of reduced complexity \(0 < \tilde{c}_4(M) < 6\). The unique closed connected 4–manifold of reduced complexity 6 is the complex projective plane \(\mathbb{C}P^2\).

(b) Let \(M^4\) be a closed connected 4–manifold. If \(\tilde{c}_4(M) = 8\), then \(M\) is (PL) homeomorphic to either \(S^1 \times S^3\) or \(S^1 \times S^3\). There are no closed connected 4–manifolds of reduced complexity 10.

(c) The unique closed connected prime 4–manifold of reduced complexity 12 is the topological product \(S^2 \times S^2\).

(d) The unique closed connected prime 4–manifold of reduced complexity 14 is the real projective 4–space \(\mathbb{R}P^4\).

Since the \(k\)–simplexes of \(K(G)\) are differentiated by their colors, the complexities were refined in [4] as follows. Given a crystallization \((G, c_G)\) of \(M\), let \(\beta_\Gamma(G) = \text{card} \, S_{\Delta_n \setminus \Gamma}(G)\) denote the number of \(k\)–simplexes of \(K(G)\) whose vertices are labeled by the colors of \(\Gamma \subset \Delta_n\), \(\text{card} \, \Gamma = k + 1\). The colored \(k\)–complexity of \(M\), \(0 \leq k \leq n\), is defined by

\[
\rho_k(M) = \min \{ \beta_\Gamma(G) : (G, c_G) \text{ is a crystallization of } M, \ \Gamma \subset \Delta_n, \ \text{card} \, \Gamma = k + 1 \}
\]

and the reduced colored \(k\)–complexities are

\[
\tilde{\rho}_k(M) = \rho_k(M) - 1 \quad 0 \leq k \leq n - 1
\]

\[
\tilde{\rho}_n(M) = \rho_n(M) - 2.
\]
Theorem 7.3 [4]

i) \( \left( \frac{n+1}{k+1} \right) \rho_k(M) \leq c_k(M) \), and equality holds for \( k = n, n - 1 \)

ii) \( 0 = \rho_0 \leq \rho_1 \leq \cdots \leq \rho_n \)

iii) The reduced colored complexities are subadditive

iv) \( rk(M^n) \leq \tilde{\rho}_1(M^n) \leq g(M^n) \) for \( n \geq 4 \), and \( \tilde{\rho}_1(M^3) = g(M^3) \) is the Heegaard genus of \( M \)

v) For \( k = n, n - 1 \) and \( n - 2 \) if \( n \geq 3 \), \( \tilde{\rho}_k(M^n) = 0 \) if and only if \( M^n \cong S^n \).

As a consequence of (iv) we have the following conjecture (which is true for \( n \leq 3 \)) (see [4], p.324)

Conjecture C(12). \( M^n \) is simply-connected if and only if \( \tilde{\rho}_1(M^n) = 0 \).

Theorem 7.4 [21]

i) For any closed connected orientable surface \( M \),

\[
2\tilde{\rho}_1(M) = \tilde{\rho}_2(M) = 4 - 2\chi(M) = 4g(M).
\]

ii) If \( M \) is a closed connected orientable 3–manifold, then

\[
\tilde{\rho}_1(M) = g(M) \quad \text{and} \quad 6g(M) \leq 2\tilde{\rho}_2(M) = \tilde{\rho}_3(M).
\]

iii) If \( M \) is a closed connected orientable 4–manifold, then

\[
\tilde{\rho}_1(M) \leq \min\{g(M), (3g(M) + \tilde{\rho}_4(M))/4\}/5\\
\tilde{\rho}_2(M) \leq (2g(M) + \tilde{\rho}_4(M))/5\\
\tilde{\rho}_3(M) = 2\tilde{\rho}_4(M)
\]

This result give rise to some interesting classes of closed orientable 3–manifolds. We say that a closed connected orientable 3–manifold \( M^3 \) is minimal if \( \tilde{\rho}_3(M) = 6g(M^3) \). For example, if \( g(M^3) = 1 \), then \( \tilde{\rho}_3(M) = 6 \) (and \( \rho_3(M) = 8 \)) hence \( M \) is PL homeomorphic to either the real projective 3–space \( \mathbb{R}P^3 \) or \( S^1 \times S^2 \).

Problem 7.2 Classify all the minimal 3–manifolds of genus greater than one.
In dimension 4, we say that $M^4$ is minimal if $\tilde{\rho}_1(M) = g(M)$ and $\tilde{\rho}_4(M) = 8g(M)$.

**Problem 7.3** Classify all the minimal 4–manifolds.

**8. Average order**

If $K$ is a simplicial triangulation of a closed connected 3–manifold $M$ with $E_0(K)$ edges and $F_0(K)$ triangles, then the average edge order of $K$ was defined in [42] as

$$\mu_0(K) = \frac{3F_0(K)}{E_0(K)}$$

The relations between this quantity and the topology of $M$ were investigated in the quoted paper, and the main result is

**Theorem 8.1** [42] Let $K$ be any simplicial triangulation of a closed connected 3–manifold $M$. Then

(a) $3 \leq \mu_0(K) < 6$, equality holds if and only if $K$ is the simplicial triangulation of the boundary of a 4–simplex.

(b) For any $\epsilon > 0$ there are simplicial triangulations $K_1$ and $K_2$ of $M$ such that

$$\mu_0(K_1) < 4.5 + \epsilon \quad \text{and} \quad \mu_0(K_2) > 6 - \epsilon.$$ (c) If $\mu_0(K) < 4.5$, then $K$ is a simplicial triangulation of $S^3$.

(d) If $\mu_0(K) = 4.5$, then $K$ is a simplicial triangulation of $S^3$, $S^1 \times S^2$ or $S^1 \times \sim S^2$.

This concept was extended in [8] to higher dimension, and successively investigated there (see [5] for the 3–dimensional case) for the class of colored triangulations of $n$–manifolds. Let now $K$ be a colored triangulation of a closed connected $n$–manifold $M$, that is, $K$ is a pseudocomplex triangulating $M$, whose vertices are labeled by $\Delta_n$ so that the coloring is injective on each $n$–simplex of $K$. For such a colored triangulation $K$, it is natural to define the average $(n - 2)$–simplex order of $K$ (see [8]) as

$$\mu(K) = \frac{n\beta_{n-1}(K)}{\beta_{n-2}(K)}$$

where $\beta_k(K)$ is the number of $k$–simplexes of $K$, $0 \leq k \leq n$. The following is the main theorem of [8] (which extends that of [5] obtained in dimension 3).
Theorem 8.2 [5] [8] Let $K$ be any colored triangulation of a closed PL $n$–manifold $M^n$, $n \geq 3$. Then

(a) $2 \leq \mu(K) < 6$, equality holds if and only if $K$ is the standard (two $n$–simplexes) colored triangulation of $\mathbb{S}^n$.

(b) For any $\epsilon > 0$ there exists a colored triangulation $K_\epsilon$ of $M$ such that

$$\mu(K_\epsilon) < \frac{2(n+1)}{(n-1)} + \epsilon.$$ If $n = 3$, there exists a colored triangulation $\overline{K}_\epsilon$ of $M$ such that $\mu(\overline{K}_\epsilon) > 6 - \epsilon$.

(c) If $\mu(K) < \frac{2(n+1)}{(n-1)}$, then $K$ is a colored triangulation of $\mathbb{S}^n$.

(d) For $3 \leq n \leq 5$, if $\mu(K) = \frac{2(n+1)}{(n-1)}$, then $K$ is a colored triangulation of one of the following $n$–manifolds: $\mathbb{S}^n$, $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ or (for $n = 3$) the real projective space $\mathbb{RP}^3$.

Theorem 8.3 Let $K$ be any contracted triangulation of a closed connected prime orientable $3$–manifold $M$. Then

(a) If $4 < \mu(K) < 5$, then $K$ is a colored triangulation of $L(3,1)$, $L(4,1)$, $L(5,2)$, or the quaternionic space $\mathbb{S}^3/\langle 2,2,2 \rangle$.

(b) If $\mu(K) = 5$, then $M$ is (PL) homeomorphic to $L(5,1)$, $L(7,2)$, $L(8,3)$, or the prism manifold $\mathbb{S}^3/\langle 3,2,2 \rangle$.

Proof. If $p$ denotes the number of tetrahedra in $K$, then $\beta_2(K) = 2p$. Since $\chi(M) = 0$ and $\beta_0(K) = 4$, we get $\beta_1(K) = 4 + \beta_2(K) - p = 4 + p$. This gives $\mu(K) = \frac{3\beta_2(K)}{\beta_1(K)} = \frac{6p}{4 + p}$. If $4 < \mu(K) < 5$, then $8 < p < 20$. Now statement (a) follows from the classification of all closed connected prime orientable $3$–manifolds which admit a contracted triangulation consisting of $18$ tetrahedra at most (see [40], Section 5.4, p.224). If $\mu(K) = 5$, then $p = 20$. Now (b) follows from [40], Section 5.4, p.225. $\square$

As done in Theorem 8.3, one can obtain the classification of closed connected prime orientable $3$–manifolds $M$ which have contracted triangulations $K$ with $\mu(K) \leq 5.25$ (use again [40] since $p \leq 28$).
Problem 8.1 Classify the topological structure of closed connected orientable 3–manifolds which admit colored triangulations $K$ with a fixed value of $\mu(K)$ such that 

$$5.25 < \mu(K) < 6.$$ 

We can define the following new concepts. Let $K$ be a colored triangulation of a closed connected $n$–manifold $M$. The average $k$–simplex order of $K$ is defined as

$$\mu_k(K) = \frac{n}{k+1} \beta_{n-1}(K) / \beta_k(K)$$

where $\beta_k(K)$ is the number of $k$–simplexes of $K$, $0 \leq k \leq n-2$. For $k = n-2$, we get the average $(n-2)$–simplex order of [8]. The average $k$–order of $M$ is defined by

$$\mu_k(M) = \min \{ \mu_k(K) : K \text{ is a colored triangulation of } M \}.$$ 

the contracted average $k$–order of $M$ is

$$\tilde{\mu}_k(M) = \min \{ \mu_k(K) : K \text{ is a contracted triangulation of } M \}.$$ 

Of course $\mu_k(M) \leq \tilde{\mu}_k(M)$, $0 \leq k \leq n$.

In general the average $\{h,k\}$–order of $M$ is

$$\mu_{h,k}(M) = \min \left\{ \frac{\left( \frac{h+1}{k+1} \right) \beta_h(K)}{\beta_k(K)} : K \text{ is a colored triangulation of } M \right\},$$

$0 \leq k < h \leq n$, and the contracted average $\{h,k\}$–order of $M$ is

$$\tilde{\mu}_{h,k}(M) = \min \left\{ \frac{\left( \frac{h+1}{k+1} \right) \beta_h(K)}{\beta_k(K)} : K \text{ is a contracted triangulation of } M \right\},$$

In particular, for $n = 3$, $h = 1$, $k = 0$ we have

$$\mu_{1,0}(K) = \frac{2\beta_1(K)}{\beta_0(K)} = \frac{2E_0(K)}{V_0(K)}.$$
which is the average vertex degree defined in [42].

**Problem 8.2** Study relations between these invariants and the topology of manifolds. Relate them to the invariants of the previous sections.

**Acknowledgements.** Work performed under the auspices of the GNSAGA of the CNR (National Research Council) of Italy and partially supported by the MIUR (Ministero dell’Istruzione, dell’Università e della Ricerca) of Italy within the project “Proprietà Geometriche delle Varietà Reali e Complesse”, and by a research grant of the University of Modena and Reggio Emilia.

**References**


Received: November, 2009