On the 2-Absorbing Ideals

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1. Introduction

The concept of 2-absorbing ideals, in a commutative ring, was introduced by A. Badawi, in [1], as a generalization of prime ideals, and some properties of 2-absorbing ideals were studied. Precisely, a proper ideal $I$ of $R$ is said to be 2-absorbing if $abc \in I$ for $a, b, c \in R$ implies that $ab \in I$ or $bc \in I$ or $ac \in I$. This paper is concerned with the development of the theory of this topic. We characterize 2-absorbing ideals of $R \times R'$, where $R$ and $R'$ are commutative rings with identity. Particularly, we obtain the relationship between 2-absorbing ideals of $R$ and $R[x]$ the polynomials ring in an indeterminate $x$ over $R$.

Throughout this paper $R$ denotes a commutative ring with non-zero identity. For an ideal $I$ of $R$ let $I :_R x$ denote the set $\{r \in R : rx \in I\}$. We say that $p \in \text{Spec}(R)$ is an associated prime ideal of $R/I$ if there exists a non-zero element $x \in R$ with $I :_R x = p$, the set of associated prime ideals of $R/I$ is
denoted by Ass$_R(R/I)$, the set of zero-divisors of $R$ is denoted by $Z_R(R)$ and all other notations are standard.

**Theorem 1.1.** Let $R'$ be a commutative ring with identity and $\varphi : R \to R'$ a ring homomorphism. If $I'$ is a 2-absorbing ideal of $R'$, then $\varphi^{-1}(I')$ is a 2-absorbing ideal of $R$. Furthermore, if $\varphi$ is an epimorphism and $I$ is a 2-absorbing ideal of $R$ containing $\text{Ker} \varphi$, then $\varphi(I)R'$ is a 2-absorbing ideal of $R'$.

**Proof.** The first assertion is obvious. For the second, assume that $a', b', c' \in R'$ and $a'b'c' \in \varphi(I)R'$. Then there are $a, b, c \in R$ such that $\varphi(a) = a'$, $\varphi(b) = b'$, $\varphi(c) = c'$ and $\varphi(abc) = a'b'c' = \sum_{i=1}^{n} \varphi(a_i)r'_i$, where $a_1, \cdots, a_n \in I$ and $r'_1, \cdots, r'_n \in R'$. By assumption there are $r_1, \cdots, r_n \in R$ such that $\varphi(r_i) = r'_i$, for $i = 1, \cdots, n$. Now, we have $\varphi(abc) = \sum_{i=1}^{n} \varphi(a_i)\varphi(r_i)$ and so $abc - \sum_{i=1}^{n} a_i r_i \in \text{Ker} \varphi \subseteq I$. Thus $abc \in I$ and $I$ is 2-absorbing so that $ab \in I$ or $ac \in I$ or $bc \in I$. If, for example, $ab \in I$, then $\varphi(ab) = a'b' \in \varphi(I)R'$ and this means that $\varphi(I)R'$ is 2-absorbing. \hfill \square

Assume that $R'$ is a commutative ring with identity, in the following we characterize the 2-absorbing ideals of the ring $R \times R'$, and we show that the set of 2-absorbing ideals of $R \times R'$ is larger than the set of prime ideals of it.

**Theorem 1.2.** If $a$ (res. $b$) is a 2-absorbing ideal of $R$ (res. $R'$), then $a \times R'$ (res. $R \times b$) is a 2-absorbing ideal of $R \times R'$. Furthermore, if $I$ is a 2-absorbing ideal of $R \times R'$, then either $I = a \times R'$ (res. $I = R \times b$), where $a$ (res. $b$) is a 2-absorbing ideal of $R$ (res. $R'$) or $I = p \times q$, where $p$ (res. $q$) is a prime ideal of $R$ (res. $R'$).

**Proof.** It is clear that $a \times R'$ (res. $R \times b$) is a 2-absorbing ideal of $R \times R'$ whenever $a$ (res. $b$) is a 2-absorbing ideal of $R$ (res. $R'$). Assume that $I$ is a 2-absorbing ideal of $R \times R'$ thus either $r(I) = P$ whit $P^2 \subseteq I$ or $r(I) = P \cap Q$ whit $PQ \subseteq I$, where $P$ and $Q$ are prime ideals of $R \times R'$ by Theorem 2.4, in [1].
If \( r(I) = P \), then either \( P = p \times R' \) or \( P = R \times q \), where \( p \) and \( q \) are prime ideals of \( R \) and \( R' \) respectively, by [2]. Let \( r(I) = P = p \times R' \). Then it is easy to see that \( a = \{ r \in R : (r, r') \in I \mbox{ for some } r' \in R' \} \) is an ideal of \( R \) and \( I \subseteq a \times R' \). Suppose that \((a, b) \in a \times R'\), thus there exists \( r' \in R' \) such that \((a, r') \in I \). Now \( p^2 \times R' \subseteq I \) implies that \((a, 0) \in I \) and consequently \((a, b) \in I \). Hence \( I = a \times R' \) and by 1.1 \( a \) is a 2-absorbing ideal of \( R \).

If \( r(I) = P \cap Q, P = p \times R' \) and \( Q = R \times q' \), where \( p \) and \( q \) are prime ideals of \( R \) and \( R' \) respectively, then \( PQ = p \times q' \subseteq I \subseteq r(I) = p \times q \). Thus \( I = p \times q \) which is a 2-absorbing ideal of \( R \times R' \). In the other cases we do the same as above.

\[ \square \]

**Theorem 1.3.** Let \( I \) be an ideal of \( R \), \( S \) a multiplicatively closed subset of \( R \) and \( S^{-1}R \) the ring of fractions of \( R \). If \( I \) is 2-absorbing ideal of \( R \) and \( S \cap I = \emptyset \), then \( S^{-1}I \) is a 2-absorbing ideal of \( S^{-1}R \). Furthermore, if \( S^{-1}I \) is 2-absorbing and \( S \cap Z_R(R/I) = \emptyset \), then \( I \) is 2-absorbing.

**Proof.** Assume that \( a, b, c \in R, s, t, l \in S \) and \( abc/stl \in S^{-1}I \). So there exists \( s' \in S \) such that \( s'abc \in I \). Thus \( s'ab \in I \) or \( s'ac \in I \) or \( bc \in I \). If \( bc \in I \), then \( bc/tl \in S^{-1}I \) and we are done. If, for example, \( s'ab \in I \), then \( s'ab/s'tl = ab/st \in S^{-1}I \). Hence, \( S^{-1}I \) is 2-absorbing. Now, assume that \( a, b, c \in R \) and \( abc \in I \). Then \( abc/1 \in S^{-1}I \). Thus \( ab/1 \in S^{-1}I \) or \( ac/1 \in S^{-1}I \) or \( bc/1 \in S^{-1}I \). If, for example, \( ab/1 \in S^{-1}I \), then there exists \( s \in S \) such that \( sab \in I \). Thus \( ab \in I \) since \( S \cap Z_R(R/I) = \emptyset \), so that \( S^{-1}I \) is 2-absorbing. \[ \square \]

**Theorem 1.4.** Let \( I \) be a 2-absorbing ideal and \( p, q \) be two prime ideals of \( R \). Then

(i) If \( r(I) = p \), then \( I :_R x \) is a 2-absorbing ideal of \( R \), for all \( x \in R \setminus p \) with \( r(I :_R x) = p \) and \( \Sigma = \{ I :_R x \mid x \in R \} \) is a totally ordered set.

(ii) If \( r(I) = p \cap q \), then \( I :_R x \) is a 2-absorbing ideal of \( R \), for all \( x \in R \setminus p \cup q \) with \( r(I :_R x) = p \cap q \) and \( \Sigma = \{ I :_R x \mid x \in R \setminus p \cup q \} \) is a totally ordered set.
(iii) If $r(I) = p \cap q$, then $I :_R x = q$, for all $x \in p \setminus q$ and $I :_R x = p$, for all $x \in q \setminus p$.

**Proof.** (i) Assume that $x \in R \setminus p$, $a, b, c \in R$ and $abc \in I :_R x$. Thus $abcx \in I$ and $I$ is a 2-absorbing ideal of $R$. So that $ax \in I$ or $bcx \in I$ or $abc \in I$. If $ax \in I$ or $bcx \in I$ we are done. If $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$ which imply that $abx \in I$ or $acx \in I$ or $bcx \in I$ that is the claim. Hence, $I :_R x$ is a 2-absorbing ideal of $R$ and it is easy to see that $I \subseteq I :_R x \subseteq p$. Suppose that $x, y \in R \setminus p$. It is clear that $xy \in R \setminus p$, $I :_R x \subseteq I :_R xy, I :_R y \subseteq I :_R xy$ and $(I :_R x) \cup (I :_R y) \subseteq I :_R xy$. To establish the reverse inclusion, let $z \in I :_R xy$. Then $xyz \in I$ which implies that either $xz \in I$ or $yz \in I$ since $xy \notin I$. Thus $I :_R xy \subseteq (I :_R x) \cup (I :_R y)$ and therefore $I :_R xy = (I :_R x) \cup (I :_R y)$. Hence, either $I :_R xy = I :_R x$ or $I :_R xy = I :_R y$. Now, either $I :_R x \subseteq I :_R y$ or $I :_R y \subseteq I :_R x$ which shows that $\Sigma = \{I :_R x \mid x \in R \setminus p\}$ is a totally ordered set. On the other hand, Theorem 2.5, in [1], shows that $I :_R x$ is a prime ideal of $R$ containing $p$, for all $x \in p \setminus I$ and $\Sigma = \{I :_R x \mid x \in p \setminus I\}$ is totally ordered. Therefore, $\Sigma = \{I :_R x \mid x \in R\}$ is totally ordered.

(ii) By a similar argument to that of (i) we can show that $I :_R x$ is a 2-absorbing ideal of $R$, $I :_R x \subseteq p \cap q$ and $r(I :_R x) = p \cap q$. Also, it is easy to see that $\Sigma = \{I :_R x \mid x \in R \setminus p \cup q\}$ is a totally ordered set.

(iii) Assume that $r(I) = p \cap q$ and $x \in p \setminus q$. If $y \in p \cap q$, then $xy \in pq \subseteq I$. Hence, $y \in I :_R x$ and so $p \cap q \subseteq I :_R x$. Also, $I :_R x \subseteq q$. Now, suppose that $y \in q$. Then $xy \in pq \subseteq I$ and it follows that $y \in I :_R x$. Hence, $q = I :_R x$. 

**Theorem 1.5.** Let $I$ be a 2-absorbing ideal and $p, q$ be two prime ideals of $R$. Then

(i) If $r(I) = p$, then $\text{Ass}_R(R/I)$ is a totally ordered set.

(ii) If $r(I) = p \cap q$, then $\text{Ass}_R(R/I)$ is union of two totally ordered set.

**Proof.** (i) Let $q' \in \text{Ass}_R(R/I)$. Then there exists $x \in R \setminus I$ such that $q' = I :_R x$. If $x \notin p$, then it is easy to see that $q' = p$. Otherwise, we have $p \subseteq q'$ and $\text{Ass}_R(R/I)$ is a totally ordered set by Theorem 2.5, in [1].
(ii) Let \( q' \in \text{Ass}_R(R/I) \). Then there exists \( x \in R \setminus I \) such that \( q' = I :_R x \). Suppose \( x \not\in p \cap q \) we show that either \( q' = p \) or \( q' = q \). We have \( r(I) = p \cap q \subseteq q' \), thus either \( p \subseteq q' \) or \( q \subseteq q' \). Assume that \( p \subseteq q' \). We show that \( x \not\in p \). If \( x \in p \), then \( x \in q' \). Thus \( x^2 \in I \subseteq q \) so that \( x \in q \). Hence, \( x \in p \cap q \) which is a contradiction. Now, let \( a \in q' \). Thus \( ax \in I \subseteq p \), and so \( a \in p \) which implies that \( q' \subseteq p \). Therefore, \( p = q' \). If \( x \in p \cap q \setminus I \), then, by Theorem 2.6 in [1], we have \( p \subseteq q' \) and \( q \subseteq q' \). Hence, \( \text{Ass}_R(R/I) \) is union of two totally ordered set. \( \square \)

**Theorem 1.6.** Let \( I \) be a 2-absorbing ideal of \( R \), \( p, q \) be two prime ideals of \( R \) and \( f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x] \). Then

(i) If \( f(x) \in r(IR[x]) \), then \( IR[x] :_{R[x]} f(x) = (I :_R a_t)R[x] \), for some \( 0 \leq t \leq n \), is a prime ideal of \( R[x] \).

(ii) If \( f(x) \not\in r(IR[x]) \), then either \( IR[x] :_{R[x]} f(x) = (I :_R a_t)R[x] \), for some \( 0 \leq t \leq n \), or \( IR[x] :_{R[x]} f(x) = pR[x] \cap qR[x] \).

**Proof.** First of all we show that \( IR[x] :_{R[x]} f(x) = \bigcap_{i=0}^n(I :_R a_i)R[x] \). It is clear that \( \bigcap_{i=0}^n(I :_R a_i)R[x] \subseteq IR[x] :_{R[x]} f(x) \) so suffice it to show that \( IR[x] :_{R[x]} f(x) \subseteq \bigcap_{i=0}^n(I :_R a_i)R[x] \). Assume that \( g(x) = b_0 + b_1x + \cdots + b_mx^m \in IR[x] :_{R[x]} f(x) \). Thus \( f(x)g(x) = \sum_{k=0}^{n+m} c_kx^k \in IR[x] \), where \( c_k = \sum_{i=0}^k a_ib_{k-i} \), for all \( 0 \leq k \leq n + m \). We have to show that \( b_k \in \bigcap_{i=0}^n(I :_R a_i) \), for all \( 0 \leq k \leq m \).

(i) It is clear that \( b_0 \in I :_R a_0 \), so assume, inductively, that \( 0 < t \leq n \) and \( b_0 \in \bigcap_{i=0}^{t-1}(I :_R a_i) \). By \( c_t = a_0b_t + \cdots + a_{t-1}b_1 + a_tb_0 \in I \) we have \( c_tb_0 + I = a_tb_0^2 + I = 0 \), which implies that \( b_0 \in I :_R a_t \) since \( I :_R a_t \) is prime, see [1] Theorems 2.5 and 2.6. Now, assume, inductively, that \( 0 < k \leq m \) and \( b_0, \cdots, b_{k-1} \in \bigcap_{i=0}^n(I :_R a_i) \). We show that \( b_k \in \bigcap_{i=0}^n(I :_R a_i) \). The induction hypothesis shows that \( c_k + I = a_0b_k + I = 0 \), thus \( b_k \in I :_R a_0 \). Assume that \( b_k \in I :_R a_j \), for all \( 0 \leq j \leq i-1 \). Hence, \( c_{k+i} + I = a_0b_{k+i} + a_1b_{k+i-1} + \cdots + a_ib_{k+1} + I = 0 \) and so \( c_{k+i}b_k + I = a_ib_{k+i}^2 + I = 0 \). Thus \( b_k \in I :_R a_i \) since \( I :_R a_i \) is prime. Therefore, by induction \( b_k \in \bigcap_{i=0}^n(I :_R a_i) \). This complete the inductive
step, and the proof of (i) is completed by another using of Theorems 2.5 and 2.6, in [1]. Therefore, $\bigcap_{i=0}^{n}(I \colon_R a_i)R[x] = (I \colon_R a_i)R[x]$, for some $0 \leq t \leq n$.

(ii) We show that $b_k \in I :_R a_0$, for all $0 \leq k \leq m$. If $r(I) = p$ and $a_0 \in p$, then by $f(x)g(x) \in pR[x]$ and $f(x) \not\in p[x]$ it follows that $g(x) \in pR[x]$. So that $b_k \in p$, for all $0 \leq k \leq m$. Hence, $a_0b_k \in p^2 \subseteq I$ which shows that $b_k \in I :_R a_0$, for all $0 \leq k \leq m$. Let $r(I) = p \cap q$ and $a_0 \in p \cap q$. Then by $f(x) \not\in p[x] \cap q[x]$ we can assume that $f(x) \in p[x]$ and $f(x) \not\in q[x]$. Thus $b_k \in p$, for all $0 \leq k \leq m$ and so $a_0b_k \in pq \subseteq I$ which shows that $b_k \in I :_R a_0$, for all $0 \leq k \leq m$. Now, assume that $a_0 \not\in r(I)$. It is clear that $b_0 \in I :_R a_0$. So assume, inductively, that $0 < t \leq m$ and $b_0, \ldots, b_{t-1} \in I :_R a_0$. By $c_t = a_0b_t + a_1b_{t-1} + \cdots + a_nb_{t-1} \in I$ we have $c_ta_0 + I = b_0a_0^2 + I = 0$, then $b_t \in I :_R a_0$ otherwise $a_0 \in r(I)$ which is a contradiction. Hence, $b_k \in I :_R a_0$, for all $0 \leq k \leq m$. Now by inductive and a similar argument to that of $a_0$ we can show that $b_k \in I :_R a_t$, for all $0 \leq k \leq m$. Hence, $IR[x] :_{R[x]} f(x) \subseteq \bigcap_{i=0}^{n}(I :_R a_i)R[x]$. If $r(I) = p$, then $\bigcap_{i=0}^{n}(I :_R a_i)R[x] = (I :_R a_i)R[x]$ by 1.4(i). If $r(I) = p \cap q$ and there exists $0 \leq t \leq n$ such that $a_t \not\in p \cup q$, then $\bigcap_{i=0}^{n}(I :_R a_i)R[x] = (I :_R a_i)R[x] \subseteq pR[x] \cap qR[x]$ by 1.4(ii). If for all $0 \leq i \leq n$ we have $a_i \in p \setminus q$, then $pR[x] \cap qR[x] \subseteq \bigcap_{i=0}^{n}(I :_R a_i)R[x] = qR[x]$ by 1.4(iii). If there exists $0 \leq k < n$ such that $a_0, \ldots, a_k \in p \setminus q$ and $a_{k+1}, \ldots, a_n \in q \setminus p$, then $\bigcap_{i=0}^{n}(I :_R a_i)R[x] = pR[x] \cap qR[x]$ by 1.4(iii). □

**Corollary 1.7.** Let $R[x]$ be the ring of polynomials in an indeterminate $x$ which coefficients in $R$. If $I$ is a 2-absorbing ideal of $R$, then $IR[x]$ is a 2-absorbing ideal of $R[x]$.

**Proof.** When $r(I) = I$ there is nothing to prove since either $IR[x] = pR[x]$ or $IR[x] = pR[x] \cap qR[x]$ which is a 2-absorbing ideal of $R[x]$. Thus we may assume that $r(I) \neq I$. Now, the assertion follows by [1] Theorems 2.8 and 2.9 and 1.6(i). □

**Corollary 1.8.** Let $I$ be a 2-absorbing ideal of $R$. Then $\text{Ass}(R[x]/IR[x])$ is a totally ordered set or is a union of two totally ordered set. Furthermore, if
On the 2-absorbing ideals

$P \in \text{Ass}(R[x]/IR[x])$, then there exists $a \in R$ such that $P = (I :_R a)R[x]$ and $P$ is a minimal element of $\text{Ass}(R[x]/IR[x])$ if and only if $a \in R \setminus r(I)$

Proof. The first assertion follows by 1.7 and 1.5. For the second, by assumption, there exists $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x]$ such that $P = IR[x] :_{R[x]} a_0 + a_1 x + \cdots + a_n x^n = \bigcap_{i=0}^n (I :_R a_i)R[x]$. Hence, 1.6 and this fact that $P$ is irreducible ideal imply that $P = (I :_R a)R[x]$.

References


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