A Note on the Density of Certain Sets of Positive Integers

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In memory of my sister Fedra Marina Jakimczuk (1970-2010)

Abstract

In former articles we have obtained a set \( A \) of positive integers with positive density \( \sigma \) and a partition of \( A \) in infinite sets \( A_i \) \((i = 1, 2, \ldots)\) with positive density \( \sigma_i \) such that the following equation holds \( \sum_{i=1}^{\infty} \sigma_i = \sigma \). Consequently the sum of the densities of the infinite sets \( A_i \) equals the density of the union \( A \) of these infinite sets. In this note we give examples where the following inequality holds \( \sum_{i=1}^{\infty} \sigma_i < \sigma \). That is, the sum of the densities of the infinite sets \( A_i \) is less than the density of the union \( A \) of these infinite sets.

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1 Introduction

In this section \( p_n \) denotes the \( n \)-th prime number. Then \( p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, \ldots \).

If \( A \) is a set of infinite positive integers and \( A(x) \) is the number of positive integers in \( A \) that do not exceed \( x \) the density of the set \( A \) is

\[
\lim_{x \to \infty} \frac{A(x)}{x},
\]

when this limit exists. Clearly the density is a nonnegative real number less than or equal to 1.

Consider the following two examples.
Example 1.1 Let $\beta_{ph}$ be the set of all positive integers whose prime factorization is of the form $p_h^{s_1} p_{h+1}^{s_2} \ldots$ where $s_i \geq 0$ ($i = h + 1, h + 2, \ldots$) and $s_h \geq 1$. That is, the set $\beta_{ph}$ of all positive integers such that the minimum prime factor in their prime factorization is $p_h$. Note that if $i \neq j$ then the sets $\beta_{pi}$ and $\beta_{pj}$ are disjoint. On the other hand $\bigcup_{h=1}^{\infty} \beta_{ph} = N - \{1\}$, where $N$ is the set of all positive integers. That is, the sets $\beta_{ph}$ ($h = 1, 2, \ldots$) are a partition of $N - \{1\}$. The density of $N - \{1\}$ is 1. In [2] is proved that the set $\beta_{ph}$ ($h = 1, 2, \ldots$) has positive density

$$D_{ph} = \left( \prod_{i=1}^{h-1} \left( 1 - \frac{1}{p_i} \right) \right) \frac{1}{p_h},$$

and that the sum of the infinite positive densities is 1. That is,

$$\sum_{h=1}^{\infty} D_{ph} = 1.$$

Consequently the sum of the densities of the infinite sets $\beta_{ph}$ equals the density of the union of these infinite sets.

Example 1.2 A positive integer $n$ is quadratfrei if it is either a product of different primes or 1. For example, $n = 2$ and $n = 5.7.23$ are quadratfrei. Let $Q_1$ be the set of quadratfrei numbers, it is well-known [1, Chapter XVIII, Theorem 333] that this set has positive density $\frac{6}{\pi^2}$. That is, if $Q_1(x)$ is the number of quadratfrei numbers not exceeding $x$ we have

$$\lim_{x \to \infty} \frac{Q_1(x)}{x} = \frac{6}{\pi^2}.$$

Let $Q_2$ be the set of not quadratfrei numbers. That is, the set of numbers such that in their prime factorization there exists a prime with exponent greater than 1. The density of this set will be $1 - \frac{6}{\pi^2}$. That is, if $Q_2(x)$ is the number of not quadratfrei numbers not exceeding $x$ we have

$$\lim_{x \to \infty} \frac{Q_2(x)}{x} = 1 - \frac{6}{\pi^2}.$$

Let us consider the set $\beta_{ph}$ of all positive integers such that in their prime factorization $p_h$ is the minimum prime with exponent greater than 1. Note that if $i \neq j$ then the sets $\beta_{pi}$ and $\beta_{pj}$ are disjoint. On the other hand $\bigcup_{h=1}^{\infty} \beta_{ph} = Q_2$. That is, the sets $\beta_{ph}$ ($h = 1, 2, \ldots$) are a partition of $Q_2$. The density of $Q_2$ is (see above)$1 - \frac{6}{\pi^2}$. In [3] is proved that the set $\beta_{ph}$ ($h = 1, 2, \ldots$) has positive density

$$D_{ph} = \left( \prod_{i=1}^{h-1} \left( 1 - \frac{1}{p_i} \right) \right) \frac{1}{p_h^2}.$$
and that the sum of the infinite positive densities is $1 - \frac{6}{\pi^2}$. That is,

$$\sum_{h=1}^{\infty} D_{ph} = 1 - \frac{6}{\pi^2}.$$  

Consequently the sum of the densities of the infinite sets $\beta_{ph}$ equals the density of the union of these infinite sets.

In these two examples we have a set $A$ of positive integers with positive density $\sigma$ and a partition of $A$ in infinite sets $A_i$ ($i = 1, 2, \ldots$) with positive density $\sigma_i$ such that the following equation holds

$$\sum_{i=1}^{\infty} \sigma_i = \sigma.$$  \hspace{1cm} (1)

Consequently the sum of the densities of the infinite sets $A_i$ equals the density of the union $A$ of these infinite sets.

2 Main Results

We have the following theorem.

**Theorem 2.1** Let $A$ be a set of positive integers with density $\sigma$ and consider a partition of $A$ in infinite sets $A_i$ ($i = 1, 2, \ldots$) with density $\sigma_i$. The following inequality holds

$$\sum_{i=1}^{\infty} \sigma_i \leq \sigma.$$  \hspace{1cm} (2)

Proof. We have for all $k \geq 1$ the inequality

$$A_1(x) + A_2(x) + \cdots + A_k(x) \leq A(x) \quad (x \geq 1),$$

where $A_i(x)$ is the number of numbers in the set $A_i$ that do not exceed $x$ and $A(x)$ is the number of numbers in the set $A$ that do not exceed $x$.

Consequently

$$\lim_{x \to \infty} \left( \frac{A_1(x) + A_2(x) + \cdots + A_k(x)}{x} \right) = \sigma_1 + \sigma_2 + \cdots + \sigma_k \leq \lim_{x \to \infty} \frac{A(x)}{x} = \sigma$$

Therefore since the $\sigma_i \geq 0$ the series $\sum_{i=1}^{\infty} \sigma_i$ is convergent and $\sum_{i=1}^{\infty} \sigma_i \leq \sigma$. The theorem is proved.

In Examples 1.1 and 1.2 equation (2) becomes the equality

$$\sum_{i=1}^{\infty} \sigma_i = \sigma.$$
However there exist examples where
\[ \sum_{i=1}^{\infty} \sigma_i < \sigma. \] (3)

We now give two examples where (3) is fulfilled.

**Example 2.2** Let \( \pi_i \) be the set of quadratfrei with \( i \) prime factors \( (i \geq 1) \) and let \( \pi_i(x) \) be the number of quadratfrei with \( i \) prime factors that do not exceed \( x \). We have [1, Chapter XXII, Theorem 437]

\[ \pi_i(x) \sim \frac{x (\log \log x)^{i-1}}{(i-1)! \log x} \quad (i \geq 1). \]

Consequently
\[ \sigma_i = \lim_{x \to \infty} \frac{\pi_i(x)}{x} = 0. \]

That is, the set \( \pi_i \ (i \geq 1) \) has density \( \sigma_i = 0 \). On the other hand the union of the sets \( \pi_i \ (i \geq 1) \) is the set \( Q_1 \) of quadratfrei numbers whose density is (see Example 1.2) \( \sigma = \frac{6}{\pi^2} \). Consequently in this example we have
\[ \sum_{i=1}^{\infty} \sigma_i = 0 < \sigma = \frac{6}{\pi^2}. \]

**Example 2.3** Let us consider the set (see Example 1.2 and Example 2.2) \( A_i = \pi_i \cup \beta_{p_i} \ (i \geq 1) \). This set has positive density \( \sigma_i = 0 + D_{p_i} = D_{p_i} \). On the other hand, the union of the sets \( A_i \ (i \geq 1) \) is the set of all positive integers whose density is \( \sigma = 1 \). Consequently in this example we have
\[ \sum_{i=1}^{\infty} \sigma_i = \sum_{i=1}^{\infty} D_{p_i} = 1 - \frac{6}{\pi^2} < \sigma = 1. \]

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**References**


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